

{ 钱学森 }

{ 力学手稿 }

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出版前言

2011年12月11日是西安交通大学杰出校友钱学森先生的百年诞辰。为缅怀钱学森学长,学习他的科学思想和卓越风范,展示其丰功伟绩和人格魅力,西安交通大学举办了“纪念钱学森诞辰100周年”系列活动:作为制片方之一,参与西部电影集团摄制传记故事片《钱学森》;与中央电视台合作,出品纪录片《实验班的故事——沿着钱学森走过的路》;扩建钱学森生平业绩展馆,向校内外开放;举办钱学森科学与教育思想研讨会;出版发行《钱学森力学手稿》、《钱学森年谱(初编)》、《钱学森第六次产业革命思想探微丛书》等。

钱学森先生在美国深造和工作期间留下大量珍贵手稿,这些手稿真实展示了钱学森先生博大精深的学识、开拓求实的精神和严谨奋进的作风,是钱老勇攀科学高峰和严谨治学的集中体现。这里,我们将部分原稿整理汇集成册,出版《钱学森力学手稿》,作为钱老百年诞辰的献礼。

《钱学森力学手稿》共10卷,包含两部分内容。第一部分是草稿,包括扁壳、球壳和圆柱壳屈曲分析的公式推导和数值演算。在研究圆柱壳轴压屈曲问题时,为了求得圆柱壳体的临界压力,在有关的五百多页草稿中,对多达二十多种可能的屈曲模

态逐一进行公式推演和数值计算,最终才找到满意的并在论文中采用的屈曲模态。仔细观察草稿中的数据列表,每个数字有效位数都长达八位,在手摇机械式计算机作为主要计算工具的年代,这串串数字凝聚着多少现今难以想象的艰辛劳动。

第二部分是手稿,以航空航天工程为核心,涵盖空气动力学、固体力学、火箭技术、工程控制论和物理力学等领域的部分学术论文手稿、打印稿和讲义。

《钱学森力学手稿》是在西安交通大学校领导的大力支持下,由西安交通大学航天航空学院沈亚鹏教授整理完成。图书出版过程中得到了西安交通大学党委宣传部、校友关系发展部、图书馆、航天航空学院等的积极协助,在此深表感谢。

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Section 1

The Buckling of Spherical Shells by External Pressure

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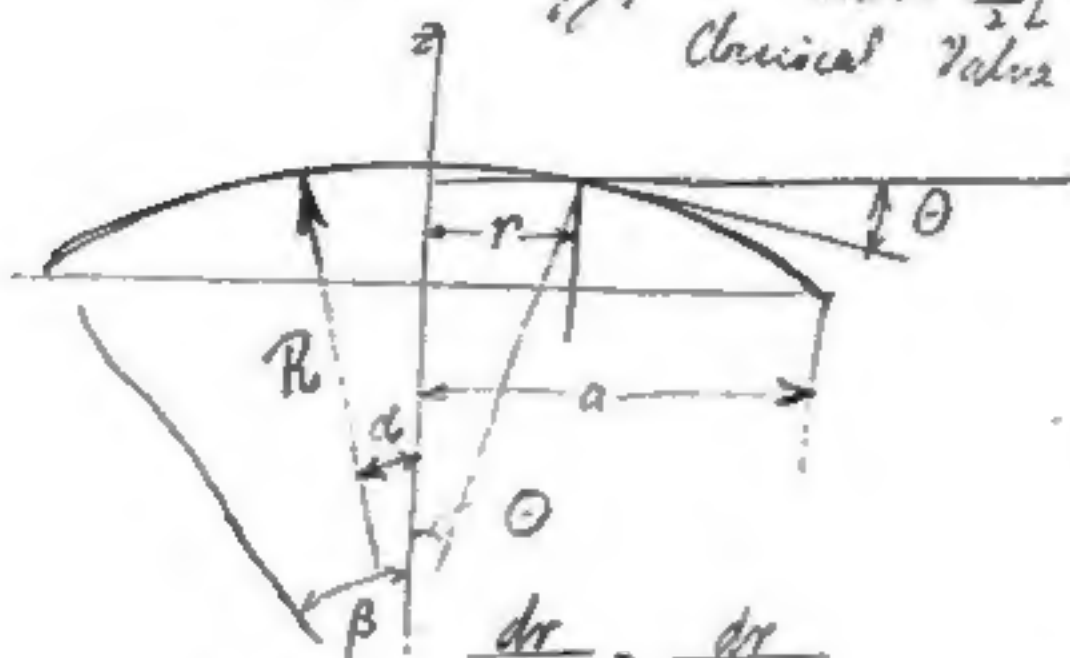
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Lab.
Tech.

$$\text{Bending Energy/unit area} = \frac{D}{2} [(k_1 + k_2)^2 - 2(1-\mu)(k_1 k_2 - z^2)] \quad (163)$$

$$\text{Extensional Energy/unit area} = \frac{Et}{2} [(\epsilon_1 + \epsilon_2)^2 - 2(1-\mu)\epsilon_1 \epsilon_2 - \frac{1}{4}\pi^2]$$

$$\text{Classical Value} = \phi = \frac{\sigma}{E(\frac{t}{R})} = \frac{1}{\sqrt{3(1-\mu^2)}} \approx 0.6$$

the original element



$$= \frac{dr}{\cos \theta_0}$$

the deflected element

$$= \frac{dr}{\cos \theta}$$

(164)

$$\epsilon = \frac{\frac{dr}{\cos \theta} - \frac{dr}{\cos \theta_0}}{\frac{dr}{\cos \theta_0}} = \frac{\cos \theta_0}{\cos \theta} - 1$$

The extensional energy

$$\frac{Et}{2} \int_0^a \left(\frac{\cos \theta_0}{\cos \theta} - 1 \right)^2 2\pi r \frac{dr}{\cos \theta_0}$$

$$= \frac{Et}{2} \int_0^\beta \left(\frac{\cos \theta_0}{\cos \theta} - 1 \right)^2 2\pi R \sin \alpha R d\alpha$$

$$= \frac{ER^3}{2} \left(\frac{t}{R} \right) \cdot 2\pi \int_0^\beta \left(\frac{\cos \theta_0}{\cos \theta} - 1 \right)^2 \sin \alpha d\alpha$$

$$\text{The curvature in meridional direction} = \frac{d\theta}{d\bar{s}} = \frac{d\theta}{d\alpha} / \frac{d\bar{s}}{d\alpha}$$

$$d\bar{s} = \frac{dr}{\cos \theta} = \frac{d[R \sin \alpha]}{\cos \theta} = \frac{R \cos \alpha d\alpha}{\cos \theta}$$

$$\text{Change in curvature} = \frac{\frac{d\theta}{d\alpha}}{\frac{R \cos \alpha}{\cos \theta}} - \frac{1}{R} = \frac{1}{R} \left[\frac{\cos \theta}{\cos \alpha} \frac{d\theta}{d\alpha} - 1 \right]$$

change in curvature in other direction

$$= \left(\frac{1}{\frac{R \sin \alpha}{\sin \Theta}} \right) - \frac{1}{R} = \frac{1}{R} \left[\frac{\sin \Theta}{\sin \alpha} - 1 \right]$$

The bending energy

$$= \frac{E t^3}{24} \int_0^\beta 2\pi R^2 \sin \alpha \, d\alpha \frac{1}{R^2} \left[\left(\frac{\cos \Theta}{\cos \alpha} \frac{d\Theta}{d\alpha} - 1 \right)^2 + \left(\frac{\sin \Theta}{\sin \alpha} - 1 \right)^2 \right]$$

$$= \frac{R^3 E}{24} \frac{(t^3)}{12} (2\pi) \int_0^\beta \sin \alpha \left[\left(\frac{\cos \Theta}{\cos \alpha} \frac{d\Theta}{d\alpha} - 1 \right)^2 + \left(\frac{\sin \Theta}{\sin \alpha} - 1 \right)^2 \right] d\alpha$$

The potential energy $\int_0^a 2\pi r z \, dr$

$$= \int_0^a \left[2\pi \frac{r^2}{2} z \right]_0^a - \int_0^a 2\pi \frac{r^2}{2} \frac{dz}{dr} dr$$

$$= - \int_0^a \pi R^2 \sin^2 \alpha \tan \Theta R \cos \alpha \, d\alpha$$

$$= - \int_0^a R^3 \pi \sin^2 \alpha \tan \Theta \cos \alpha \, d\alpha$$

$$E\left(\frac{t}{R}\right) \int_0^{\theta} \left(\frac{\cos \alpha}{\cos \theta} - 1\right)^2 \sin \alpha d\alpha + \left[\frac{E\left(\frac{t}{R}\right)^3}{12} \int_0^{\theta} \left\{ \left(\frac{\cos \theta}{\cos \alpha} \frac{d\theta}{d\alpha} - 1\right)^2 + \left(\frac{\sin \theta}{\sin \alpha} - 1\right)^2 \right\} \sin \alpha d\alpha \right] \quad (85)$$

$$+ p \int_0^{\theta} \sin^2 \alpha \cos \alpha \tan \theta d\alpha$$

The differential equation

$$E\left(\frac{t}{R}\right) \left[2 \sin \alpha \left(\frac{\cos \alpha}{\cos \theta} - 1\right) \frac{\cos \alpha}{\cos^2 \theta} \sin \theta \right]$$

$$+ \frac{E\left(\frac{t}{R}\right)^3}{12} \left[2 \sin \alpha \left\{ -\left(\frac{\cos \theta}{\cos \alpha} \frac{d\theta}{d\alpha} - 1\right) \frac{\sin \theta}{\cos \alpha} \frac{d\theta}{d\alpha} + \left(\frac{\sin \theta}{\sin \alpha} - 1\right) \frac{\cos \theta}{\sin \alpha} \right\} \right.$$

$$\left. - \frac{d}{d\alpha} \left\{ 2 \sin \alpha \left(\frac{\cos \theta}{\cos \alpha} \frac{d\theta}{d\alpha} - 1\right) \frac{\cos \theta}{\cos \alpha} \right\} \right]$$

$$+ p \left[\sin^2 \alpha \cos \alpha \sec^2 \theta \right] = 0$$

$$2E\left(\frac{t}{R}\right) \left[\left(\frac{\cos \alpha}{\cos \theta} - 1\right) \frac{\sin \alpha \cos \alpha}{\cos \theta} \tan \theta \right]$$

$$+ \frac{E\left(\frac{t}{R}\right)^3}{6} \left[\sin \alpha \left\{ \left(\frac{\sin \theta}{\sin \alpha} - 1\right) \frac{\cos \theta}{\sin \alpha} - \left(\frac{\cos \theta}{\cos \alpha} \frac{d\theta}{d\alpha} - 1\right) \frac{\sin \theta}{\cos \alpha} \frac{d\theta}{d\alpha} \right\} \right.$$

$$\left. - \left\{ \sec^2 \alpha \cos \theta \left(\frac{\cos \theta}{\cos \alpha} \frac{d\theta}{d\alpha} - 1\right) - \tan \alpha \sin \theta \left(\frac{\cos \theta}{\cos \alpha} \frac{d\theta}{d\alpha} - 1\right) \frac{d\theta}{d\alpha} \right. \right.$$

$$\left. \left. + \tan \alpha \cos \theta \left(\frac{-\sin \theta}{\cos \alpha} \left(\frac{d\theta}{d\alpha}\right)^2 + \frac{\cos \theta \tan \alpha}{\cos \alpha} \left(\frac{d\theta}{d\alpha}\right) + \frac{\cos \theta}{\cos \alpha} \frac{d^2 \theta}{d\alpha^2} \right) \right\} \right]$$

$$+ p \left[\sin^2 \alpha \cos \alpha \sec^2 \theta \right] = 0.$$

$$2E\left(\frac{t}{R}\right) \left[\frac{\sin \alpha \cos \alpha}{\cos \Theta} \tan \Theta \left(\frac{\cos \alpha}{\cos \Theta} - 1 \right) \right] \quad (16)$$

$$+ \frac{E\left(\frac{t}{R}\right)^3}{6} \left[\cos \Theta \left(\frac{\sin \Theta}{\sin \alpha} + 2 \tan^2 \alpha \right) - \frac{\cos^3 \Theta}{\cos \alpha} (2 \tan^2 \alpha + 1) \frac{d\Theta}{d\alpha} + \frac{\sin \Theta \cos \Theta \tan \alpha}{\cos \alpha} \left(\frac{d\Theta}{d\alpha} \right)^2 \right. \\ \left. - \frac{\cos^3 \Theta \tan^3 \alpha}{\cos \alpha} \frac{d^2 \Theta}{d\alpha^2} \right] + \sin^2 \alpha \cos \alpha \sec^3 \Theta \phi = 0$$

The other boundary condition: if the support is a hinge support is

$$\begin{aligned} & \text{at } \alpha = \beta \quad \tan \Theta \left(\frac{\cos \Theta}{\cos \alpha} \frac{d\Theta}{d\alpha} - 1 \right) = 0 \quad \text{at } \alpha = \beta \\ & \text{or} \quad \frac{\cos \Theta}{\cos \alpha} \frac{d\Theta}{d\alpha} = 1 \quad \text{at } \alpha = \beta \\ & \quad \quad \quad [\text{no change in curvature}] \end{aligned} \quad \left. \begin{array}{l} \text{the other} \\ \text{boundary condition} \\ \text{is} \\ \Theta = 0 \text{ at } \alpha = 0 \end{array} \right\}$$

If we neglect the bending energy, then

$$2E\frac{t}{R} \left[\sin \Theta \left(\frac{\cos \alpha}{\cos \Theta} - 1 \right) \right] + \sin \alpha \phi = 0$$

Let Θ_1 be the value of Θ at $\alpha = \beta$.

$$2E\frac{t}{R} \left[\sin \Theta_1 \left(\frac{\cos \beta}{\cos \Theta_1} - 1 \right) \right] + \sin \beta \phi = 0$$

$$\cancel{\left(E\frac{t}{R} \right) \phi = - \left(\frac{\cos \beta}{\cos \Theta_1} - 1 \right) \frac{\sin \Theta_1}{\sin \beta}}$$

$$\frac{\phi}{E\left(\frac{t}{R}\right)} = - 2 \frac{\sin \Theta_1}{\sin \beta} \left(\frac{\cos \beta}{\cos \Theta_1} - 1 \right)$$

$$\frac{\partial \phi}{\partial \Theta_1} = 0 \quad \text{gives} \quad \cos \beta = \cos^3 \Theta_1 ;$$

$$\dots \frac{f}{E(\frac{f}{k})} = -2 \frac{[1 - (\cos \beta)^{2/3}]^{3/2}}{\sin \beta} [(\cos \beta)^{2/3} - 1]$$

$$= 2 \frac{[1 - (\cos \beta)^{2/3}]^{3/2}}{\sin \beta}$$

187)

When β is small, $1 - (\cos \beta)^{2/3} = \frac{1}{3} \beta^2$

$$\frac{f}{E(\frac{f}{k})} \approx 2 \frac{1}{(3)^{3/2}} \beta^2 = 0.385 \beta^2$$

When $\beta = 0.1$ $\frac{f}{E(\frac{f}{k})} \approx 0.00385$ $f \approx 0.00385 E(\frac{f}{k})$

The critical stress = $\frac{f \cdot r}{2t} = 0.001925 E$

Experiments = $\frac{0.25 \times 0.60}{900} E = 0.000167 E$ for $\frac{A}{t} = 300$,

If $\beta = 0.0294$, the values will agree.

The necessary section to make a 1.5" radius dent will be

51"

From the relation

$$\sin \Theta \left(\frac{\cos \alpha}{\cos \Theta} - 1 \right) + \frac{f}{2} \sin \alpha \left(\frac{r}{E k} \right) = 0$$

$$\sin \Theta \cos \alpha - \sin \Theta \cos \Theta + \frac{f}{2} k \cdot \sin \alpha \cos \Theta = 0.$$

$$(\cos \Theta \cos \alpha - \cos 2\Theta - \frac{f}{2} k \cdot \sin \alpha \sin \Theta) \frac{d\Theta}{d\alpha} - \sin \Theta \sin \alpha + \frac{f}{2} \cos \alpha \cos \Theta = 0.$$

$$\frac{d\Theta}{d\alpha} = \frac{\sin \Theta \sin \alpha - \frac{f}{2} k \cos \alpha \cos \Theta}{\cos \Theta \cos \alpha - \cos 2\Theta - \frac{f}{2} k \sin \alpha \sin \Theta}$$

If Θ, α are small, then

188)

$$\frac{1}{2}\Theta(\Theta^2 - \alpha^2) + \frac{1}{2}h\alpha = 0$$

$$\text{or } \Theta^3 - \alpha^2\Theta + h\alpha = 0$$

$$\therefore \Theta = \left(-\frac{h\alpha}{2} + \sqrt{\frac{h^2\alpha^2}{4} - \frac{\alpha^6}{27}}\right)^{\frac{1}{3}} + \left(-\frac{h\alpha}{2} - \sqrt{\frac{h^2\alpha^2}{4} - \frac{\alpha^6}{27}}\right)^{\frac{1}{3}}$$

$$= \alpha^{\frac{1}{3}} \left[\left(-\frac{h}{2} + \sqrt{\frac{h^2}{4} - \frac{\alpha^4}{27}}\right)^{\frac{1}{3}} + \left(-\frac{h}{2} - \sqrt{\frac{h^2}{4} - \frac{\alpha^4}{27}}\right)^{\frac{1}{3}} \right]$$

etc.

$$\left(\frac{\alpha \cdot \Theta}{\Theta^3} - 1\right) = -\frac{1}{2} \frac{h}{\Theta^2}$$

$$\frac{d\Theta}{d\alpha} = \frac{2\Theta\alpha - h}{3\Theta^2 - \alpha^2 - h\Theta\alpha}$$

If the Θ, α are of same order and are small, the differential equation can be written as

$$\underbrace{E\left(\frac{1}{h}\right) \alpha \Theta(\Theta^2 - \alpha^2) + \alpha^2 \beta}_{\text{etc.}} + \frac{E\left(\frac{1}{h}\right)^2}{6} \left\{ \left(\frac{\Theta}{\alpha} + \alpha^2\right) - (2\alpha^2 + 1) \frac{d\Theta}{d\alpha} + \alpha \Theta \left(\frac{d\Theta}{d\alpha}\right)^2 - \alpha \frac{d^2\Theta}{d\alpha^2} \right\} = 0$$

or approximately

$$\alpha \Theta(\Theta^2 - \alpha^2) + h_1 \alpha^2 + h_2 \left\{ \frac{\Theta}{\alpha} - \frac{d\Theta}{d\alpha} - \alpha \frac{d^2\Theta}{d\alpha^2} \right\} = 0.$$

$$\text{where } h_1 = \frac{h}{E\left(\frac{1}{h}\right)}, \quad h_2 = \frac{\left(\frac{1}{h}\right)^2}{6}$$

If α, Θ are small, the total energy can be written as (89)

$$E\left(\frac{f}{k}\right) \int_0^1 \frac{1}{4} (\Theta^2 - \alpha^2)^2 \alpha d\alpha + \frac{E\left(\frac{f}{k}\right)^3}{12} \int_0^1 \left\{ \left(\frac{d\Theta}{d\alpha} - 1 \right)^2 + \left(\frac{\Theta}{\alpha} - 1 \right)^2 \right\} \alpha d\alpha + p \int_0^1 \alpha^2 \Theta d\alpha$$

Let $\Theta = C\alpha$, then we have

$$E\left(\frac{f}{k}\right) \frac{1}{4} (C^2 - 1)^2 \int_0^1 \alpha^5 d\alpha + \frac{E\left(\frac{f}{k}\right)^3}{12} (C-1)^2 \int_0^1 2\alpha d\alpha + pC \int_0^1 \alpha^3 d\alpha$$

$$= E\left(\frac{f}{k}\right) \frac{1}{24} (C^2 - 1)^2 \beta^6 + \frac{E\left(\frac{f}{k}\right)^3}{12} (C-1)^2 \beta^2 + \frac{pC}{4} \beta^4$$

Differentiate with respect to C ,

$$\frac{E\left(\frac{f}{k}\right)}{6} C(C^2 - 1) \beta^2 + \frac{E\left(\frac{f}{k}\right)^3}{6} \frac{(C-1)}{\beta^2} + \frac{p}{4} = 0$$

Or
$$p = \frac{2}{3} E\left(\frac{f}{k}\right) \left\{ C(1 - C^2) \beta^2 + \left(\frac{f}{k}\right)^2 \frac{(1-C)}{\beta^2} \right\}$$

Minimizing with respect to β^2 ,

$$C(1 - C^2) \beta^2 = \left(\frac{f}{k}\right)^2 \frac{(1-C)}{\beta^2}$$

or
$$\beta^2 = \left(\frac{f}{k}\right) \sqrt{\frac{1-C}{C(1-C^2)}} = \left(\frac{f}{k}\right) \frac{1}{C^{1/2}(1+C)^{1/2}}$$

$$p = \frac{2}{3} E\left(\frac{f}{k}\right) \left\{ 2\left(\frac{f}{k}\right) C^{1/2} (1-C)(1+C)^{1/2} \right\}$$

$$= \frac{4}{3} E\left(\frac{f}{k}\right)^2 \left\{ C^{1/2} (1+C)^{1/2} (1-C) \right\}$$

Minimizing with respect to C , $\frac{\partial}{\partial C} [C - C^2 - C^3 + C^4] = 0$

or
$$1 - 2C - 3C^2 + 4C^3 = 0$$

Divide through by $C-1$,

$$4C^2 + C - 1 = 0$$

$$C = \frac{1}{8}[-1 \pm \sqrt{1+16}]$$

$$= \frac{1}{8}[-1 \pm \sqrt{17}] = \begin{matrix} +0.3906 \\ -0.6406 \end{matrix}$$

$$C^{\frac{1}{2}}(1+C)^{\frac{1}{2}}(1-C) = 0.3906^{\frac{1}{2}} \times 1.3906^{\frac{1}{2}} \times 0.6096 = 0.625 \times 1.129 \times 0.6096 \\ = 0.450$$

$$\rho = 0.64E \left(\frac{t}{R}\right)^2$$

$$\sigma = 0.30E \left(\frac{t}{R}\right)$$

$$\rho^2 = \left(\frac{t}{R}\right)^2 \frac{1}{0.625 \times 1.129} \\ = 1.357 \left(\frac{t}{R}\right)^2$$

$$\text{or } \left(\frac{t}{R}\right)^2 = \frac{1}{1.357}, \quad \rho^2 = 0.001567 \\ \rho = \underline{\underline{0.0396}}$$

Let us put $\Theta = C \left[\alpha + \left(\frac{1}{C} - 1\right) \frac{\alpha^2}{\rho} \right]$, when $\alpha = \rho$, $\Theta = \rho$.

Then $\Theta^2 - \alpha^2 = \alpha^2 \left\{ C^2 \left[1 + \left(\frac{1}{C} - 1\right) \frac{\alpha}{\rho} \right]^2 - 1 \right\}$

$$\frac{d\Theta}{d\alpha} = C \left[1 + 2 \left(\frac{1}{C} - 1\right) \frac{\alpha}{\rho} \right]$$

Thus put this value to the integral for potential energy,

$$\begin{aligned}
& \frac{E(\frac{t}{R})}{4} \int_0^{\beta} \left\{ C^2 \left[1 + \left(\frac{1}{C} - 1 \right) \left(\frac{\alpha}{\beta} \right) \right]^2 - 1 \right\}^2 \alpha^5 d\alpha + \frac{E(\frac{t}{R})^3}{12} \int_0^{\beta} \left\{ \left[(C-1) + 2(1-C) \left(\frac{\alpha}{\beta} \right) \right]^2 \left[(C-1) + (1-C) \left(\frac{\alpha}{\beta} \right) \right] \right\} \alpha d\alpha \\
& + \beta C \int_0^{\beta} \left\{ \alpha^3 + \left(\frac{1}{C} - 1 \right) \frac{\alpha^4}{\beta} \right\} d\alpha \\
& = \frac{E(\frac{t}{R})}{4} \int_0^{\beta} \left\{ (C^2-1)^2 + 4C^2 \left(\frac{1}{C} - 1 \right) (C^2-1) \left(\frac{\alpha}{\beta} \right) + 2C^2 (3C^2-1) \left(\frac{1}{C} - 1 \right) \left(\frac{\alpha}{\beta} \right)^2 \right. \\
& \quad \left. + 4C^4 \left(\frac{1}{C} - 1 \right)^3 \left(\frac{\alpha}{\beta} \right)^3 + C^2 \left(\frac{1}{C} - 1 \right)^4 \left(\frac{\alpha}{\beta} \right)^4 \right\} \alpha^5 d\alpha \\
& + \frac{E(\frac{t}{R})^3}{12} (C-1)^2 \int_0^{\beta} \left\{ 2 - 6 \left(\frac{\alpha}{\beta} \right) + 5 \left(\frac{\alpha}{\beta} \right)^2 \right\} \alpha d\alpha + \beta C \int_0^{\beta} \left\{ \alpha^3 + \left(\frac{1}{C} - 1 \right) \frac{\alpha^4}{\beta} \right\} d\alpha \\
& = \frac{E(\frac{t}{R})}{4} \int \left\{ \frac{(C^2-1)^2}{6} + \frac{4}{7} C(1-C)(C^2-1) + \frac{1}{4} (3C^2-1)(1-C)^2 + \frac{4}{9} C(1-C)^3 + \frac{1}{10} (1-C)^4 \right\} \beta^6 \\
& + \frac{E(\frac{t}{R})^3}{12} (C-1)^2 \frac{\beta^2}{4} + \beta C \left\{ \frac{1}{4} + \frac{1}{5} \left(\frac{1}{C} - 1 \right) \right\} \beta^4
\end{aligned}$$

Minimizing with respect to C ,

$$\begin{aligned}
& \frac{E(\frac{t}{R})}{4} \left\{ \frac{2}{3} C(C^2-1) - \frac{4}{7} (C-1)(4C^2+C-1) + \frac{1}{2} (C-1)(6C^2-3C-1) - \frac{4}{9} (C-1)(4C^2-5C+1) \right. \\
& \quad \left. + \frac{2}{5} (C-1)^2 \right\} \beta^6 \\
& + \frac{E(\frac{t}{R})^3}{12} \frac{(C-1)\beta^2}{2} + \beta \left\{ \frac{1}{4} - \frac{1}{5} \right\} \beta^4 = 0 \\
& \beta = 5 E(\frac{t}{R}) \left[(1-C) \beta^2 \left\{ \frac{2}{3} C(C+1) - \frac{4}{7} (4C^2+C-1) + \frac{1}{2} (6C^2-3C-1) - \frac{4}{9} (4C^2-5C+1) \right. \right. \\
& \quad \left. \left. + \frac{2}{5} (C-1)^2 \right\} \right. \\
& \quad \left. + \left(\frac{t}{R} \right)^2 \frac{(1-C)}{6\beta^2} \right]
\end{aligned}$$

Minimizing with respect to ρ^2

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$$\rho^2 = \frac{(\frac{t}{R})^2}{\sqrt{6}} \left\{ \right\}^{\frac{1}{2}}$$

$$\therefore \rho = \frac{10 (\frac{t}{R})^2 E}{\sqrt{6}} \left[(1-c) \left\{ \frac{2}{3} C(C+1) - \frac{4}{7} (4C^2+C-1) + \frac{1}{2} (6C^2-3C-1) - \frac{1}{7} (4C^2-5C+1) + \frac{2}{5} (C-1)^2 \right\}^{\frac{1}{2}} \right]$$

$$= \frac{5 E (\frac{t}{R})^2}{3 \sqrt{105}} (1-c) \left\{ 420(C^2+C) - 360(4C^2+C-1) + 315(6C^2-3C-1) - 280(4C^2-5C+1) + 252(C^2-2C+1) \right\}^{\frac{1}{2}}$$

$$= \frac{5}{3} \frac{E (\frac{t}{R})^2}{\sqrt{105}} (1-c) \left\{ 2C^2 + 11C + 17 \right\}^{\frac{1}{2}}$$

Minimizing $(1-c)^2 (2C^2 + 11C + 17)$

$$-2(2C^2 + 11C + 17) + (1-c)(4C + 11) = 0$$

$$\text{or } 4C^2 + 22C + 34 + 4C^2 + 11C - 4C - 11 = 0$$

$$\text{or } 8C^2 + 29C + 23 = 0$$

$$C = \frac{1}{16} \left[-29 \pm \sqrt{29^2 - 32 \times 23} \right] = \frac{1}{16} \left[-29 \pm \sqrt{105} \right]$$

$$= \frac{1}{16} \left[-29 \pm 10.25 \right] = \begin{matrix} -1.171 \\ -2.452 \end{matrix}$$

$$(2C^2 + 11C + 17) = (1-c)(2C + 5.5) = \begin{matrix} 2.271 \times 3.158 = 7.175 \\ 3.452 \times 0.596 = 2.056 \end{matrix}$$

$$(1-c)(2C^2 + 11C + 17)^{\frac{1}{2}} = \begin{matrix} 2.271 \times 2.678 = 6.08 \\ 3.452 \times 1.434 = 4.95 \end{matrix}$$

$$p = \frac{5 \times 4.95}{3 \times 10.25} E \left(\frac{t}{R} \right)^2 = \frac{0.804}{0.989} E \left(\frac{t}{R} \right)^2$$

$$\sigma = \frac{0.402}{0.499} E \left(\frac{t}{R} \right)$$

$$\sigma = k E \left(\frac{t}{R} \right)$$

$$C = 1, \quad k = 0$$

$$C = 0.5 \quad k = \frac{5}{30.25} \frac{0.5 \times 4.796}{2} = 0.1951$$

$$C = 0 \quad k = \frac{5}{30.25} \frac{17^{\frac{1}{2}}}{2} = 0.335$$

$$C = -0.5, \quad k = \frac{5}{30.25} \frac{1.5 \times 12^{\frac{1}{2}}}{2} = 0.422$$

$$C = -1.171 \quad k = 0.499$$

$$C = -1.5 \quad k = \frac{5}{30.25} \frac{2.5 \times 5^{\frac{1}{2}}}{2} = 0.455$$

$$C = -2.0 \quad k = \frac{5}{30.25} \frac{3.0 \times 3^{\frac{1}{2}}}{2} = 0.422$$

$$C = -2.452 \quad k = 0.402$$

$$C = -3 \quad k = \frac{5}{30.25} 2 \times 2^{\frac{1}{2}} = 0.460$$

$$C = -4 \quad k = \frac{5}{30.25} 2.5 \times 5^{\frac{1}{2}} = 0.909$$

for free end, $p = \frac{4}{3} E \left(\frac{1}{R} \right)^3 (c + c^2)^{\frac{1}{2}} (1 - c)$

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$$k = \frac{2}{3} (1 - c) (c + c^2)^{\frac{1}{2}}$$

$$C = 1$$

$$k = 0$$

$$C = 0.1$$

$$k = \frac{2}{3} 0.9 \cdot 0.11^{\frac{1}{2}} = 0.1991$$

$$C = 0.2$$

$$k = \frac{2}{3} 0.8 \cdot 0.24^{\frac{1}{2}} = 0.2612$$

$$C = 0.3$$

$$k = \frac{2}{3} 0.7 \cdot 0.39^{\frac{1}{2}} = 0.2915$$

$$C = 0.3904$$

$$k = 0.300$$

$$C = 0.50$$

$$k = \frac{2}{3} 0.5 \cdot 0.75^{\frac{1}{2}} = 0.2885$$

$$C = 0.60$$

$$k = \frac{2}{3} 0.4 \cdot 0.96^{\frac{1}{2}} = 0.2612$$

$$C = 0.8$$

$$k = \frac{2}{3} 0.2 \cdot 1.44^{\frac{1}{2}} = 0.1600$$

$$C = 0$$

$$k = 0$$

$$C = -1$$

$$k = 0$$

$$C = -1.1$$

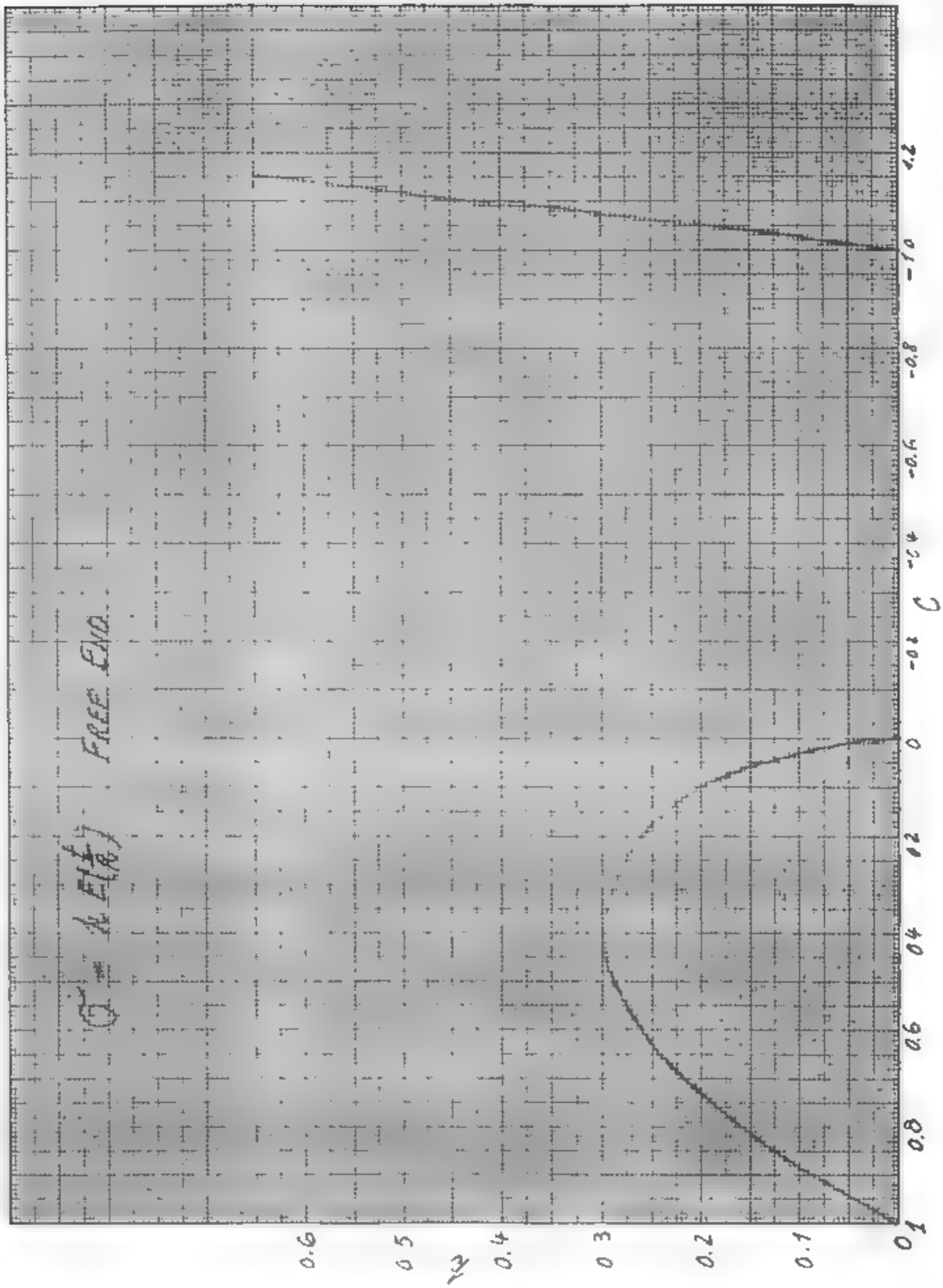
$$k = \frac{2}{3} 2.1 \cdot 0.11^{\frac{1}{2}} = 0.465$$

$$C = -1.15$$

$$k = \frac{2}{3} 2.15 \cdot 0.1225^{\frac{1}{2}} = 0.595$$

$$\frac{4.225}{1.1} = 3.8409$$

$$\frac{4.21}{1.1} = 3.8273$$



We have the differential equation

$$\alpha \Theta (\Theta^2 - \alpha^2) + k_1 \alpha^2 + k_2 \left\{ \frac{\Theta}{\alpha} - \frac{d\Theta}{d\alpha} - \alpha \frac{d^2\Theta}{d\alpha^2} \right\} = 0$$

Divide through by β^4 and write $\frac{\alpha}{\beta} \sim \alpha$, $\frac{\Theta}{\beta} \sim \Theta$,

$$K_1 = \frac{k_1}{\beta^2} = \frac{2\sigma}{E\beta^2}, \quad K_2 = \frac{\left(\frac{k_2}{\beta}\right)^2}{6\beta^4}$$

$$\alpha \Theta (\Theta^2 - \alpha^2) + K_1 \alpha^2 + K_2 \left\{ \frac{\Theta}{\alpha} - \frac{d\Theta}{d\alpha} - \alpha \frac{d^2\Theta}{d\alpha^2} \right\} = 0$$

Boundary condition for clamped end, $\Theta = 0$ when $\alpha = 0$
 $\Theta = 1$ when $\alpha = 1$.

$$\alpha^2 \frac{d^2\Theta}{d\alpha^2} + \alpha \frac{d\Theta}{d\alpha} - \Theta = \frac{1}{K_2} \alpha^2 \Theta (\Theta^2 - \alpha^2) + \frac{K_1}{K_2} \alpha^2$$

Taking the linear part

$$\alpha^2 \frac{d^2\Theta}{d\alpha^2} + \alpha \frac{d\Theta}{d\alpha} - \Theta = \frac{K_1}{K_2} \alpha^2$$

The appropriate solution is $\Theta_0 = C\alpha + \frac{K_1}{8K_2} \alpha^3$

$$\begin{aligned} \frac{1}{K_2} \alpha^2 \Theta (\Theta^2 - \alpha^2) &= \frac{1}{K_2} \alpha^4 \left(C\alpha + \frac{K_1}{8K_2} \alpha^3 \right) \left[\left(C + \frac{K_1}{8K_2} \alpha^2 \right)^2 - 1 \right] \\ &= \frac{1}{K_2} \alpha^5 \left\{ \left(C + \frac{K_1}{8K_2} \alpha^2 \right)^3 - \left(C + \frac{K_1}{8K_2} \alpha^2 \right) \right\} \\ &= \frac{1}{K_2} \alpha^5 \left[C(C^2 - 1) + (3C^2 - 1) \frac{K_1}{8K_2} \alpha^2 + 3C \left(\frac{K_1}{8K_2} \right)^2 \alpha^4 + \left(\frac{K_1}{8K_2} \right)^3 \alpha^6 \right] \\ &= \frac{1}{K_2} \left[C(C^2 - 1) \alpha^5 + (3C^2 - 1) \frac{K_1}{8K_2} \alpha^7 + 3C \left(\frac{K_1}{8K_2} \right)^2 \alpha^9 + \left(\frac{K_1}{8K_2} \right)^3 \alpha^{11} \right] \\ \Theta_1 &= C\alpha + \frac{K_1}{8K_2} \alpha^3 + \frac{1}{K_2} \alpha^5 \left[\frac{C(C^2 - 1)}{24} + (3C^2 - 1) \frac{K_1}{8K_2} \frac{\alpha^2}{48} + 3C \left(\frac{K_1}{8K_2} \right)^2 \frac{\alpha^4}{80} \right. \\ &\quad \left. + \left(\frac{K_1}{8K_2} \right)^3 \frac{\alpha^6}{120} \right] \end{aligned}$$

Write the signal equation

$$x^2 \frac{d^2 \Theta}{dx^2} + x \frac{d\Theta}{dx} - \Theta = C_1 x^3 + C_2 x^2 \Theta(\Theta^2 - x^2)$$

where $C_1 = \frac{k_1}{k_2}, \quad C_2 = \frac{1}{k_2}.$

Let $\Theta = a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7 + a_8 x^8 + \dots$

$$\begin{aligned} \Theta^2 - x^2 = & (a_1^2 - 1)x^2 + 2a_1 a_2 x^3 + (2a_1 a_3 + a_2^2)x^4 + (2a_1 a_4 + 2a_2 a_3)x^5 \\ & + (2a_1 a_5 + 2a_2 a_4 + a_3^2)x^6 + (2a_1 a_6 + 2a_2 a_5 + 2a_3 a_4)x^7 \\ & + (2a_1 a_7 + 2a_2 a_6 + 2a_3 a_5 + a_4^2)x^8 + (2a_1 a_8 + 2a_2 a_7 + 2a_3 a_6 + 2a_4 a_5)x^9 + \dots \end{aligned}$$

$$\begin{aligned} \Theta(\Theta^2 - x^2) = & a_1(a_1^2 - 1)x^3 + [2a_1^2 a_2 + a_2(a_1^2 - 1)]x^4 + [a_1(2a_1 a_3 + a_2^2) + 2a_1 a_2^2 + a_3(a_1^2 - 1)]x^5 \\ & + [a_1(2a_1 a_4 + 2a_2 a_3) + a_2(2a_1 a_3 + a_2^2) + 2a_1 a_2 a_3 + a_4(a_1^2 - 1)]x^6 + \dots \end{aligned}$$

$$C_1 x^3 + C_2 x^2 \Theta(\Theta^2 - x^2)$$

$$\begin{aligned} = & C_1 x^3 + 0 + C_2 a_1(a_1^2 - 1)x^3 + C_2 [2a_1^2 a_2 + a_2(a_1^2 - 1)]x^4 \\ & + C_2 [a_1(2a_1 a_3 + a_2^2) + 2a_1 a_2^2 + a_3(a_1^2 - 1)]x^5 \\ & + C_2 [a_1(2a_1 a_4 + 2a_2 a_3) + a_2(2a_1 a_3 + a_2^2) + 2a_1 a_2 a_3 + a_4(a_1^2 - 1)]x^6 + \dots \end{aligned}$$

$$(x^2 \frac{d^2}{dx^2} + x \frac{d}{dx} - 1) a_n x^n = (n^2 - 1) a_n x^n$$

thus

$$(a_1 = ?)$$

$$a_2 = 0$$

$$a_3 = \frac{C_1}{8}$$

$$a_4 = 0$$

$$a_5 = \frac{C_2}{24} \left[a_1 (a_1^2 - 1) \right]$$

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$$a_6 = 0$$

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$$a_7 = \frac{C_2}{48} \left[\frac{a_1^2 C_2}{4} + \frac{C_1}{8} (a_1^2 - 1) \right] - \frac{C_2 C_1}{384} (3a_1^2 - 1)$$

$$a_8 = 0.$$

$$a_9 = \frac{C_2}{80} \left[a_1 \left\{ 2a_1^2 \frac{C_2}{24} (a_1^2 - 1) + \frac{C_1^2}{64} \right\} + \frac{C_1}{8} \left\{ 2a_1 \frac{C_1}{8} + \frac{C_2}{24} a_1 (a_1^2 - 1) \right\} \right]$$

$$= \frac{C_2}{80} \left[\frac{C_2}{24} a_1 (3a_1^2 - 1)(a_1^2 - 1) + \frac{C_1^2}{64} \cdot 3a_1 \right]$$

$$= \frac{C_2}{640} \left[\frac{C_2}{3} a_1 (3a_1^2 - 1)(a_1^2 - 1) + \frac{3}{8} C_1^2 a_1 \right]$$

The condition for a_1 is [free end].

$$1 = a_1 + 2a_2 + 3a_3 + 4a_4 + 5a_5 + 6a_6 + 7a_7 + 8a_8 + 9a_9 \dots$$

$$\therefore 1 = a_1 + \frac{3}{8} C_1 + \frac{5}{24} C_2 a_1 (a_1^2 - 1) + \frac{7}{384} C_1 C_2 (3a_1^2 - 1) + \frac{9}{640} C_2 \left[\frac{C_2}{3} a_1 (3a_1^2 - 1)(a_1^2 - 1) + \frac{3}{8} C_1^2 a_1 \right]$$

+-----

This can be put into the form

$$A' C_1^2 + B' C_1 + D' = 0.$$

where $A' = \frac{27}{5120} a_1 C_2$

$$B' = \frac{3}{8} + \frac{7}{384} (3a_1^2 - 1) C_2$$

$$D' = (a_1 - 1) + \frac{5}{24} a_1 (a_1^2 - 1) C_2 + \frac{3}{640} a_1 (3a_1^2 - 1)(a_1^2 - 1) C_2^2$$

$$C_1 = \frac{\frac{25}{E \beta^2}}{\frac{(\frac{\beta}{R})^2}{6 \rho^4}} = 12 \left(\frac{\sigma}{E \frac{\beta}{R}} \right) \left(\frac{A^2}{\frac{\beta}{R}} \right) - 12 \left(\frac{\beta^2}{\frac{\beta}{R}} \right) S \quad (100)$$

$$\text{where } S = \frac{\sigma}{E \frac{\beta}{R}}$$

$$C_2 = 6 \left(\frac{\beta^2}{\frac{\beta}{R}} \right)^2$$

$$A' 144 \left(\frac{\beta^2}{\frac{\beta}{R}} \right)^2 S^2 + 12 \left(\frac{\beta^2}{\frac{\beta}{R}} \right) B' S + D' = 0$$

$$AS^2 + BS + D = 0$$

$$A = \frac{27}{5120} a_1 \left[\left(\frac{\beta^2}{\frac{\beta}{R}} \right)^2 \right]^2 144 \left[\left(\frac{\beta^2}{\frac{\beta}{R}} \right)^2 \right]^2 = \frac{729}{160} \left\{ \frac{\beta^2}{\frac{\beta}{R}} \right\}^4 a_1 = \frac{729}{160} \phi^4 a_1$$

$$B = 12 \left(\frac{\beta^2}{\frac{\beta}{R}} \right) \left[\frac{3}{8} + \frac{42}{324} (3a_1^2 - 1) \left(\frac{\beta^2}{\frac{\beta}{R}} \right) \right] = \phi \left\{ \frac{9}{4} + \frac{21}{16} (3a_1^2 - 1) \phi^2 \right\}$$

$$D = (a_1 - 1) + \frac{5}{4} a_1 (a_1^2 - 1) \phi^2 + \frac{27}{160} a_1 (3a_1^2 - 1) (a_1^2 - 1) \phi^4$$

$$\text{When } \phi = 1, \quad \beta = 0.0316, \quad \frac{\beta}{R} = \frac{1}{1000}$$

$$A = \frac{729}{160} a_1$$

$$B = \frac{9}{4} + \frac{21}{16} (3a_1^2 - 1)$$

$$D = (a_1 - 1) + \frac{5}{4} a_1 (a_1^2 - 1) + \frac{27}{160} a_1 (3a_1^2 - 1) (a_1^2 - 1)$$

$$= (a_1 - 1) \left\{ 1 + \frac{5}{4} a_1 (a_1 + 1) + \frac{27}{160} a_1 (a_1 + 1) (3a_1^2 - 1) \right\}$$

$$(1) \quad a_1 = 1.0, \quad S = 0$$

$$(2) \quad a_1 = 0.9, \quad A = 4.10 \quad B = 2.25 + \frac{2}{16} \times 1.43 = \frac{2.25}{4.127}$$

$$D = -0.1 \left[1 + \frac{5}{4} \times 0.9 \times 1.9 + \frac{2}{160} \times 0.9 \times 1.9 \times 1.43 \right]$$

$$= -0.1 (1 + 2.136 + 0.4125) = -0.3549$$

$$S = \frac{1}{8.20} \left[-4.127 \pm \sqrt{4.127^2 + 16.40 \times 0.3549} \right]$$

$$= \frac{1}{8.20} \left[-4.127 \pm 4.127 \times 1.1582 \right] = +0.0798$$

$$(3) \quad a_1 = 0.8 \quad A = 3.644, \quad B = 2.25 + \frac{2}{16} \times 0.92 = \frac{2.25}{3.457}$$

$$D = -0.2 \left[1 + 0.8 \times 1.8 \left(1.25 + \frac{2}{160} \times 0.92 \right) \right]$$

$$= -0.2 (1 + 2.024) = -0.6048$$

$$S = \frac{1}{7.288} \left[-3.457 \pm \sqrt{3.457^2 + 14.576 \times 0.6048} \right]$$

$$= \frac{3.457}{7.288} [-1 + 1.318] = 0.151$$

$$(4) \quad a_1 = 0.7 \quad A = 3.190 \quad B = 2.25 + \frac{2}{16} \times 0.47 = 2.8665$$

$$D = -0.3 \left[1 + 0.7 \times 1.7 \left(1.25 + \frac{2}{160} \times 0.47 \right) \right] = -0.774$$

$$S = \frac{1}{6.38} \left[-2.8665 \pm 2.8665 \times 1.483 \right] = +0.217$$

$$(5) \quad a_1 = 0.6 \quad A = 2.730 \quad B = 2.25 + \frac{2}{16} \times 0.08 = 2.555$$

$$D = -0.4 \left[1 + 0.6 \times 1.6 \left(1.25 + \frac{2}{160} \times 0.08 \right) \right] = -0.885$$

$$\zeta = \frac{2.355}{5.460} \times 0.655 = 0.2824$$

$$(6) \quad a_1 = 0.5 \quad \zeta = +0.359$$

$$(7) \quad a_1 = 0, \quad A=0, \quad B = \frac{15}{16}, \quad D = -1$$

$$\zeta = \frac{16}{15} = 1.066$$

for clamped end

$$1 = a_1 + \frac{C_1}{f} + \frac{C_2}{24} a_1 (a_1^2 - 1) + \frac{C_1 C_2}{384} (3a_1^2 - 1) + \frac{C_2}{640} \left[\frac{C_1}{3} a_1 (a_1^2 - 1) (3a_1^2 - 1) + \frac{3}{f} C_1^2 a_1 \right]$$

$$A' = \frac{3}{5120} a_1 C_2$$

$$B' = \frac{1}{f} + \frac{1}{384} (3a_1^2 - 1) C_2$$

$$D' = (a_1 - 1) + \frac{1}{24} a_1 (a_1^2 - 1) C_2 + \frac{1}{1920} a_1 (a_1^2 - 1) (3a_1^2 - 1) C_2^2$$

$$A = \frac{11}{160} \phi^4 a_1$$

$$B = \phi \left\{ \frac{3}{4} + \frac{3}{16} (3a_1^2 - 1) \phi^2 \right\}$$

$$D = (a_1 - 1) + \frac{1}{4} a_1 (a_1^2 - 1) \phi^2 + \frac{3}{160} a_1 (3a_1^2 - 1) (a_1^2 - 1) \phi^4$$

$$\frac{-81}{160} a_1 \frac{\phi^8}{(\frac{\phi}{r})^4} s^2 + \frac{\phi^2}{(\frac{\phi}{r})^4} \left\{ \frac{3}{4} + \frac{3}{16} (3a_1 - 1) \frac{\phi^0}{(\frac{\phi}{r})^2} \right\} s + \left\{ (a_1 - 1) + \frac{1}{4} a_1 (a_1 - 1) \frac{\phi^2}{(\frac{\phi}{r})^2} + \frac{3}{160} a_1 (3a_1 - 1) (a_1 - 1) \frac{\phi^4}{(\frac{\phi}{r})^4} \right\} s^0.$$

$$\frac{-81}{40} a_1 \frac{\phi^6}{(\frac{\phi}{r})^4} s^2 + \frac{\phi^2}{(\frac{\phi}{r})^4} \left\{ \frac{3}{4} + \frac{9}{16} (3a_1 - 1) \frac{\phi^0}{(\frac{\phi}{r})^2} \right\} s + \left\{ \frac{1}{2} a_1 (a_1 - 1) \frac{\phi^2}{(\frac{\phi}{r})^2} + \frac{3}{40} a_1 (3a_1 - 1) (a_1 - 1) \frac{\phi^4}{(\frac{\phi}{r})^4} \right\} s^0.$$

$$\frac{-81}{40} a_1 \phi^4 s^2 + \phi \left\{ 3 + \frac{3}{4} (3a_1 - 1) \phi^2 \right\} s + \left\{ 4(a_1 - 1) + a_1 (a_1 - 1) \phi^2 + \frac{3}{40} a_1 (a_1 - 1) (3a_1 - 1) \phi^4 \right\} s^0$$

$$\frac{81}{40} a_1 \phi^4 s^2 + \phi \left\{ \frac{3}{4} + \frac{9}{16} (3a_1 - 1) \phi^2 \right\} s + \left\{ \frac{1}{2} a_1 (a_1 - 1) \phi^2 + \frac{3}{40} a_1 (a_1 - 1) (3a_1 - 1) \phi^4 \right\} s^0 = 0$$

$$\phi \left\{ \frac{9}{4} + \frac{3}{16} (3a_1 - 1) \phi^2 \right\} s + \left\{ 4(a_1 - 1) + \frac{1}{2} a_1 (a_1 - 1) \phi^2 \right\} s^0 = 0.$$

$$s = - \frac{4(a_1 - 1) + \frac{1}{2} a_1 (a_1 - 1) \phi^2}{\phi \left\{ \frac{9}{4} + \frac{3}{16} (3a_1 - 1) \phi^2 \right\}}$$

Use energy method, $\Theta = C \left[a + \left(\frac{1}{c} - 1 \right) \frac{a^2}{\beta^2} \right]$

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$$\Theta^2 - a^2 = a^2 \left\{ C^2 \left[1 + \left(\frac{1}{c} - 1 \right) \left(\frac{a}{\beta} \right)^2 \right]^2 - 1 \right\}$$

$$= a^2 \left\{ (C^2 - 1) + 2C^2 \left(\frac{1}{c} - 1 \right) \left(\frac{a}{\beta} \right)^2 + C^2 \left(\frac{1}{c} - 1 \right)^2 \left(\frac{a}{\beta} \right)^4 \right\}$$

$$\frac{d\Theta}{da} = C \left[1 + 3 \left(\frac{1}{c} - 1 \right) \left(\frac{a}{\beta} \right)^2 \right]$$

$$\frac{d\Theta}{da} - 1 = (C - 1) + 3C \left(\frac{1}{c} - 1 \right) \left(\frac{a}{\beta} \right)^2 = (C - 1) \left\{ 1 + 3 \left(\frac{a}{\beta} \right)^2 \right\}$$

$$\frac{\Theta}{a} - 1 = (C - 1) \left\{ 1 + \left(\frac{a}{\beta} \right)^2 \right\}$$

The total potential energy

$$\frac{E \left(\frac{1}{R} \right)}{4} \int_0^\beta \left\{ (C^2 - 1) + 4C^2 \left(\frac{1}{c} - 1 \right) \left(\frac{a}{\beta} \right)^2 + 5C^2 \left(3C^2 - 1 \right) \left(\frac{1}{c} - 1 \right)^2 \left(\frac{a}{\beta} \right)^4 + 6C^4 \left(\frac{1}{c} - 1 \right)^3 \left(\frac{a}{\beta} \right)^6 + C^4 \left(\frac{1}{c} - 1 \right)^4 \left(\frac{a}{\beta} \right)^8 \right\} a^5 da$$

$$+ \frac{E \left(\frac{1}{R} \right)^3}{12} (C - 1)^2 \int_0^\beta \left\{ 2 - 8 \left(\frac{a}{\beta} \right)^2 + 10 \left(\frac{a}{\beta} \right)^4 \right\} a da + \beta C \int_0^\beta \left\{ a^2 + \left(\frac{1}{c} - 1 \right) \frac{a^5}{\beta^2} \right\} dx$$

$$= \frac{E \frac{1}{R}}{4} \left\{ \frac{(C^2 - 1)^2}{6} + \frac{1}{2} C (1 - C) (C^2 - 1) + \frac{1}{5} (3C^2 - 1) (1 - C)^2 + \frac{1}{3} C (1 - C)^3 + \frac{1}{14} (1 - C)^4 \right\} \beta^6$$

$$+ \frac{E \left(\frac{1}{R} \right)^3}{12} (C - 1)^2 \frac{2}{3} \beta^3 + \beta C \left\{ \frac{1}{6} + \frac{1}{6} \left(\frac{1}{c} - 1 \right) \right\} \beta^6$$

Minimizing with respect to C

$$\frac{E \left(\frac{1}{R} \right)}{4} \left\{ \frac{2}{3} C (C^2 - 1) - \frac{1}{2} (C - 1) (4C^2 + C - 1) + \frac{2}{5} (C - 1) (6C^2 - 3C - 1) - \frac{1}{3} (C - 1) (4C^2 - 5C + 1) + \frac{2}{7} (C - 1)^3 \right\} \beta^6$$

$$+ \frac{E \left(\frac{1}{R} \right)^3}{12} (C - 1) \frac{1}{3} \beta^3 + \frac{1}{12} \beta \beta^6 = 0$$

$$\beta = 3E\left(\frac{t}{R}\right) \left[(1-c)\beta^2 \left\{ \frac{2}{3}C(C+1) - \frac{1}{5}(4C^2+C-1) + \frac{2}{5}(6C^2-3C-1) - \frac{1}{3}(4C^2-5C+1) + \frac{2}{7}(C-1)^2 \right\} + \left(\frac{t}{R}\right)^2 \frac{1}{9}(1-c) \frac{1}{\beta^2} \right] \quad 205)$$

Minimizing with respect to β^2

$$\beta^2 = \frac{2}{3} \left(\frac{t}{R}\right) \left\{ \right\}^{-\frac{1}{2}}$$

$$\beta = 4E\left(\frac{t}{R}\right)^2 \left[(1-c) \left\{ \frac{2}{3}C(C+1) - \frac{1}{5}(4C^2+C-1) + \frac{2}{5}(6C^2-3C-1) - \frac{1}{3}(4C^2-5C+1) + \frac{2}{7}(C-1)^2 \right\}^{\frac{1}{2}} \right]$$

$$= \frac{4}{\sqrt{210}} E\left(\frac{t}{R}\right)^2 (1-c) \left\{ 140(C^2+C) - 105(4C^2+C-1) + 84(6C^2-3C-1) - 70(4C^2-5C+1) + 60(C-1)^2 \right\}^{\frac{1}{2}}$$

$$= \frac{4}{\sqrt{210}} E\left(\frac{t}{R}\right)^2 (1-c) (4C^2+13C+11)^{\frac{1}{2}}$$

Minimizing $(1-c)^2(4C^2+13C+11)$

$$-2(4C^2+13C+11) + (1-c)(8C+13) = 0$$

$$\text{or } 16C^2+31C+9=0 \quad -0.3556$$

$$C = \frac{-31 \pm \sqrt{961 - 96 \times 16}}{32} = \frac{-31 \pm 19.62}{32} = \frac{-0.3556}{-1.5625} = -1.512$$

$$(4C^2 + 13C + 11) = \frac{(1-C)(8C+13)}{2} = (1-C)(4C+6.5) = \frac{1.356 \times 5.078}{2.582 \times 0.1725} \quad (206)$$

$$= 6.886$$

$$0.4454$$

$$K = (1-C)(4C^2 + 13C + 11)^{\frac{1}{2}} = \frac{1.356 \times 2.624}{2.582 \times 0.6674} = \frac{3.558}{1.723}$$

$$\beta = \frac{4}{14.49} K E \left(\frac{t}{R}\right)^2$$

$$\sigma = \frac{2}{14.49} K E \left(\frac{t}{R}\right) = E \left(\frac{t}{R}\right) \frac{0.49.11}{0.2378}$$

La!!!

$$\beta^2 = \frac{2 \times 14.49}{3 \times 0.674} \left(\frac{t}{R}\right) = \frac{2 \times 14.49}{2700 \times 0.674} = 0.01592$$

$$\beta = 0.126 \text{ Radians} = \underline{7.2^\circ}$$

$$\frac{\sigma_0}{E(t)} = \frac{2}{\sqrt{210}} (1-C) (4C^2 + 13C + 11)^{\frac{1}{2}}$$

$$= \frac{2}{14.49}$$

$$\begin{array}{r} 1756249 \\ 3512498 \\ \hline 1.6487502 \\ \hline 1.8243751 \end{array}$$

$$\text{for } \Theta = C \alpha \left[1 + \left(\frac{1}{C} - 1 \right) \left(\frac{\alpha}{\beta} \right)^2 \right]$$

(206a)

for p. 206

$\left(\frac{s}{t} \right)$	① C	② $4C^2$	③ $13C$	④ $4C^2 + 13C + 11$	⑤ $(1-C)$	⑥ $\frac{④}{⑤}$	⑦ ϕ
0	1	4	13	28	0	5.292	0
0.0985	0.8	2.56	10.4	23.96	0.2	4.900	0.1353
0.2148	0.6	1.44	7.8	20.24	0.4	4.500	0.2484
0.3530	0.4	0.64	5.2	16.44	0.6	4.105	0.3400
0.520	0.2	0.16	2.6	13.76	0.8	3.710	0.4097
0.727	0	0	0	11	1.0	3.317	0.4581
0.990	-0.2	0.16	-2.6	8.56	1.2	2.926	0.4846
1.335	-0.4	0.64	-5.2	6.44	1.4	2.538	0.4904
1.793	-0.6	1.44	-7.8	4.64	1.6	2.154	0.4757
2.44	-0.8	2.56	-10.4	3.16	1.8	1.778	0.4417
3.42	-1.0	4	-13	2.00	2.0	1.414	0.3903
4.93	-1.2	5.76	-15.6	1.16	2.2	1.077	0.3270
7.25	-1.4	7.84	-18.2	0.64	2.4	0.40	0.2650
9.47	-1.6	10.24	-20.8	0.44	2.6	0.6633	0.2380
	-1.8	12.96	-23.4	0.56	2.8	0.7483	0.2892
	-2.0	16.00	-26.0	1.00	3.0	1.0000	0.4141
	-2.2	19.36	-28.6	1.76	3.2	1.327	0.5861

$$z_0 + \int_0^\beta \frac{dz}{dr} dr = 0$$

206
b

$$z_0 = - \int_0^\beta \frac{dz}{dr} dr = + R \int_0^\beta \tan \theta \cos \alpha d\alpha$$

$$\approx + R \int_0^\beta \theta d\alpha$$

$$\theta = C \left[\alpha + \left(\frac{1}{C} - 1 \right) \frac{\alpha^3}{\beta^2} \right]$$

$$z_0 = + R \int_0^\beta \left[C \alpha + (1-C) \frac{\alpha^3}{\beta^2} \right] d\alpha$$

$$= + R \left[C \frac{\beta^2}{2} + \frac{1-C}{4} \beta^2 \right]$$

$$= + R \left[\frac{1+C}{4} \right] \beta^2$$

$$\text{Original } (z_0)_{\text{ori}} = R [1 - \cos \beta] \approx R \frac{\beta^2}{2}$$

$$\delta = R \beta^2 \left[\frac{1}{2} - \frac{1+C}{4} \right] = R \beta^2 \frac{(1-C)}{4} = \underline{\underline{z_{\text{inv}} - z_0}}$$

$$\frac{\delta}{R} = \frac{(1+C)}{4} \frac{2}{3} \left(\frac{1}{R} \right) \left\{ \frac{2}{3} C (C+1) - \frac{1}{2} (4C^2 + C - 1) + \frac{1}{5} (6C^3 - 3C - 1) - \frac{1}{3} (4C^3 - 5C + 1) + \frac{1}{7} (C-1)^2 \right\}^{\frac{1}{2}}$$

$$\text{But } \frac{1}{2} \left\{ \frac{\sigma}{E \left(\frac{r}{R} \right)} \right\} \frac{1}{1-C} = \left\{ \right\}^{\frac{1}{2}}$$

$$\frac{\delta}{R} = \left(\frac{1}{R} \right) \cdot \frac{(1-C)}{4} \frac{2}{3} \frac{2(1-C)}{\phi} = \left(\frac{1}{R} \right) \frac{(1-C)(1-C)}{3\phi} = \frac{1}{R} \frac{(1-C)^2}{3\phi}$$

$$\text{Put } \Theta = C_1 \alpha \left[1 + \left(\frac{1}{C_1} - 1 \right) \left(\frac{\alpha}{\beta} \right)^2 \right] + C_2 \alpha \left[1 + \left(\frac{1}{C_2} - 1 \right) \left(\frac{\alpha}{\beta} \right)^2 \right] \quad 202)$$

$$= \alpha \left\{ (C_1 + C_2) + (1 - C_1) \left(\frac{\alpha}{\beta} \right)^2 + (1 - C_2) \left(\frac{\alpha}{\beta} \right)^2 \right\}$$

$$\Theta^2 \alpha^2 = \alpha^2 \left\{ [(C_1 + C_2) - 1]^2 + 2(1 - C_1)(C_1 + C_2) \left(\frac{\alpha}{\beta} \right)^2 + [(1 - C_1)^2 + 2(C_1 + C_2)(1 - C_2)] \left(\frac{\alpha}{\beta} \right)^4 \right. \\ \left. + 2(1 - C_1)(1 - C_2) \left(\frac{\alpha}{\beta} \right)^6 + (1 - C_2)^2 \left(\frac{\alpha}{\beta} \right)^8 \right\}$$

$$\left\{ \Theta^2 \alpha^2 \right\}^2 = \alpha^4 \left\{ [(C_1 + C_2) - 1]^2 + 4(1 - C_1)(C_1 + C_2) \left[(C_1 + C_2) - 1 \right] \left(\frac{\alpha}{\beta} \right)^2 \right. \\ \left. + \left\{ 4(1 - C_1)^2(C_1 + C_2)^2 + 2[(C_1 + C_2) - 1][(1 - C_1)^2 + 2(C_1 + C_2)(1 - C_2)] \right\} \left(\frac{\alpha}{\beta} \right)^4 \right. \\ \left. + 4 \left\{ [(C_1 + C_2) - 1](1 - C_1)(1 - C_2) + (1 - C_1)(C_1 + C_2)[(1 - C_1)^2 + 2(C_1 + C_2)(1 - C_2)] \right\} \left(\frac{\alpha}{\beta} \right)^6 \right. \\ \left. + \left\{ 2(1 - C_2)^2[(C_1 + C_2) - 1] + 8(1 - C_1)^2(1 - C_2)(C_1 + C_2) + [(1 - C_1)^2 + 2(1 - C_2)(C_1 + C_2)]^2 \right\} \left(\frac{\alpha}{\beta} \right)^8 \right. \\ \left. + 4 \left\{ (1 - C_1)(1 - C_2)^2(C_1 + C_2) + (1 - C_1)(1 - C_2)[(1 - C_1)^2 + 2(C_1 + C_2)(1 - C_2)] \right\} \left(\frac{\alpha}{\beta} \right)^{10} \right. \\ \left. + \left\{ 2(1 - C_2)^2[(1 - C_1)^2 + 2(1 - C_2)(C_1 + C_2)] + 4(1 - C_1)^2(1 - C_2)^2 \right\} \left(\frac{\alpha}{\beta} \right)^{12} \right. \\ \left. + 4(1 - C_1)(1 - C_2)^3 \left(\frac{\alpha}{\beta} \right)^{14} + (1 - C_2)^4 \left(\frac{\alpha}{\beta} \right)^{16} \right\}$$

$$\frac{d\Theta}{d\alpha} - 1 = \left\{ [(C_1 + C_2) - 1] + 3(1 - C_1) \left(\frac{\alpha}{\beta} \right)^2 + 5(1 - C_2) \left(\frac{\alpha}{\beta} \right)^4 \right\}$$

$$\frac{\Theta}{\alpha} - 1 = \left\{ [(C_1 + C_2) - 1] + (1 - C_1) \left(\frac{\alpha}{\beta} \right)^2 + (1 - C_2) \left(\frac{\alpha}{\beta} \right)^2 \right\}$$

$$\left(\frac{d\Theta}{d\alpha} - 1 \right)^2 + \left(\frac{\Theta}{\alpha} - 1 \right)^2 = 2[(C_1 + C_2) - 1]^2 + 8(1 - C_1)[(C_1 + C_2) - 1] \left(\frac{\alpha}{\beta} \right)^2 \\ + \left\{ 10(1 - C_1)^2 + 12(1 - C_2)[(C_1 + C_2) - 1] \right\} \left(\frac{\alpha}{\beta} \right)^4 + 32(1 - C_1)(1 - C_2) \left(\frac{\alpha}{\beta} \right)^6 \\ + 26(1 - C_2)^2 \left(\frac{\alpha}{\beta} \right)^8$$

$$\begin{aligned}
\int_0^{\beta} (\Theta^2 x^2)^2 dx &= \frac{1}{6} [(C_1 + C_2)^2 - 1]^2 - \frac{1}{2} (C_1 - 1)(C_1 + C_2) [(C_1 + C_2)^2 - 1] \\
&+ \frac{1}{5} \left\{ 2(C_1 - 1)^2 (C_1 + C_2)^2 + [(C_1 + C_2)^2 - 1] [(C_1 - 1)^2 - 2(C_1 + C_2)(C_2 + 1)] \right\} \\
&+ \frac{1}{3} \left\{ [(C_1 + C_2)^2 - 1] (C_1 - 1)(C_2 + 1) - (C_1 - 1)(C_1 + C_2) [(C_1 - 1)^2 - 2(C_1 + C_2)(C_2 + 1)] \right\} \\
&+ \frac{1}{14} \left\{ 2(C_2 + 1)^2 [(C_1 + C_2)^2 - 1] - 8(C_1 - 1)^2 (C_2 + 1)(C_1 + C_2) + [(C_1 - 1)^2 - 2(C_1 + C_2)(C_2 + 1)]^2 \right\} \\
&+ \frac{1}{4} \left\{ -(C_1 - 1)(C_2 + 1)^2 (C_1 + C_2) + (C_1 - 1)(C_2 + 1) [(C_1 - 1)^2 - 2(C_1 + C_2)(C_2 + 1)] \right\} \\
&+ \frac{1}{9} \left\{ (C_2 + 1)^2 [(C_1 - 1)^2 - 2(C_1 + C_2)(C_2 + 1)] + 2(C_1 - 1)^2 (C_2 + 1)^2 \right\} \\
&+ \frac{1}{5} (C_1 - 1)(C_2 + 1)^3 + \frac{1}{2 \cdot 2} (C_2 + 1)^4 \quad \text{multiplied by } \beta^6
\end{aligned}$$

$$\begin{aligned}
\int_0^{\beta} \left\{ \left(\frac{\partial \Theta}{\partial x} - 1 \right)^2 + \left(\frac{\Theta}{x} - 1 \right)^2 \right\} x dx &= [(C_1 + C_2) - 1]^2 - 2(C_1 - 1)[(C_1 + C_2) - 1] \\
&+ \frac{1}{3} \left\{ 5(C_1 - 1)^2 - 6(C_2 + 1)[(C_1 + C_2) - 1] \right\} + 4(C_1 - 1)(C_2 + 1) + 2 \cdot 6(C_2 + 1)^2 \\
&\quad \text{multiplied by } \beta^2
\end{aligned}$$

$$\int_0^{\beta} x^2 \Theta dx = \frac{1}{4} (C_1 + C_2) - \frac{1}{6} (C_1 - 1) - \frac{1}{8} (C_2 + 1) \quad \text{multiplied by } \beta^4$$

$$\text{Put } \lambda = \frac{C_2}{C_1}$$

$$\begin{aligned}
\int_0^1 (\Theta^2 - \alpha^2) \alpha d\alpha &= \frac{1}{6} [C_1^2 (1+\lambda)^2 - 1] - \frac{1}{2} C_1 (C_1 - 1) (1+\lambda) [C_1^2 (1+\lambda)^2 - 1] \quad 209) \\
&+ \frac{1}{5} \left\{ 3 C_1^2 (C_1 - 1)^2 (1+\lambda)^2 - 2 C_1^3 (C_1, \lambda, \frac{1}{4}) (1+\lambda)^3 - (C_1 - 1)^2 + 2 C_1 (1+\lambda) (\lambda C_1, \frac{1}{4}) \right\} \\
&+ \frac{1}{3} \left\{ 3 C_1^2 (C_1 - 1) (1+\lambda)^2 (\lambda C_1, \frac{1}{4}) - (C_1 - 1) (\lambda C_1, \frac{1}{4}) - C_1 (C_1 - 1)^2 (1+\lambda) \right\} \\
&+ \frac{1}{14} \left\{ 2 (C_1, \lambda, \frac{1}{4})^2 [C_1^2 (1+\lambda)^2 - 1] - 2 C_1 (C_1 - 1)^2 (\lambda C_1, \frac{1}{4}) (1+\lambda) + [(C_1 - 1)^2 - 2 C_1 (1+\lambda) (\lambda C_1, \frac{1}{4})]^2 \right\} \\
&+ \frac{1}{4} \left\{ (C_1 - 1)^3 (\lambda C_1, \frac{1}{4}) - 3 C_1 (C_1 - 1) (1+\lambda) (\lambda C_1, \frac{1}{4})^2 \right\} + \frac{1}{9} \left\{ 3 (C_1 - 1)^3 (\lambda C_1, \frac{1}{4})^2 - 2 C_1 (1+\lambda) (\lambda C_1, \frac{1}{4})^3 \right\} \\
&+ \frac{1}{5} (C_1 - 1) (\lambda C_1, \frac{1}{4})^3 + \frac{1}{22} (\lambda C_1, \frac{1}{4})^4 \quad \text{multiplied by } \beta^6
\end{aligned}$$

$$\begin{aligned}
\int_0^1 \left[\left| \frac{d\Theta}{d\alpha} - 1 \right|^2 + \left| \frac{\Theta}{\alpha} - 1 \right|^2 \right] \alpha d\alpha &= [C_1 (1+\lambda) - 1]^2 - 2 (C_1 - 1) [C_1 (1+\lambda) - 1] \\
&+ \frac{5}{3} (C_1 - 1)^2 - 2 (\lambda C_1, \frac{1}{4}) [C_1 (1+\lambda) - 1] + 4 (C_1 - 1) (\lambda C_1, \frac{1}{4}) + \frac{13}{5} (\lambda C_1, \frac{1}{4})^2 \\
&\quad \text{multiplied by } \beta^2
\end{aligned}$$

$$\int_0^1 \alpha \Theta d\alpha = \frac{C_1}{4} (1+\lambda) - \frac{1}{6} (C_1 - 1) - \frac{1}{8} (\lambda C_1, \frac{1}{4}) \quad \text{multiplied by } \beta^2$$

$$\begin{aligned}
& \frac{2}{3}(1+\lambda)^2 [(1+\lambda)^2 C_1^3 - C_1] - \frac{1}{2}(1+\lambda) [4(1+\lambda)^2 C_1^3 - 3(1+\lambda)^2 C_1^2 - 2C_1 + 1] \\
& + \frac{2}{5} \left\{ 2[3(1+\lambda)^2 - 2\lambda(1+\lambda)^3] C_1^3 - 3[3(1+\lambda)^2 - (1+\lambda)^3] C_1^2 + [3(1+\lambda)^2 + 2\lambda(1+\lambda) - 1] C_1 \right. \\
& \quad \left. + [1 - (1+\lambda)] \right\} \\
& + \frac{1}{3} \left\{ 4[3(1+\lambda)^2 - (1+\lambda)] C_1^3 - 9[(1+\lambda)^3 - (1+\lambda)] C_1^2 + 2[3(1+\lambda)^2 - 3(1+\lambda) - \lambda] C_1 + 2(1+\lambda) \right\} \\
& + \frac{2}{7} \left\{ [6\lambda^2(1+\lambda)^2 - 12\lambda(1+\lambda) + 1] C_1^3 - [3\lambda(1+\lambda)^2 - 6(1+2\lambda)(1+\lambda) - 3\lambda + 6\lambda^2(1+\lambda)] C_1^2 \right. \\
& \quad \left. + [\lambda(1+2\lambda) - 4(1+\lambda)(2+\lambda) + (1-2\lambda)] C_1 + [2\lambda + 2(1+\lambda)] \right\} \\
& + \frac{1}{4} \left\{ 4\lambda[1 - 3\lambda(1+\lambda)] C_1^3 - 3[(1+3\lambda) - 3\lambda(1+\lambda)(2+\lambda)] C_1^2 + 2(1+\lambda)[3 - 12\lambda + 1] C_1 \right. \\
& \quad \left. + 2\lambda \right\} \\
& + \frac{2}{9} \left\{ 2\lambda^2[3 - 2\lambda(1+\lambda)] C_1^3 - 9\lambda(1+\lambda)(1-\lambda) C_1^2 + 3[1 + 2\lambda - \lambda^2] C_1 - 2(1+\lambda) \right\} \\
& + \frac{1}{5} \left\{ 4\lambda^3 C_1^3 - 3\lambda^2(3+\lambda) C_1^2 + 6\lambda(1+\lambda) C_1 - (1+3\lambda) \right\} \\
& + \frac{2}{11} \lambda \{ \lambda^3 C_1^3 - 3\lambda^2 C_1^2 + 3\lambda C_1 + 1 \} = I_1
\end{aligned}$$

$$\begin{aligned}
& 2(1+\lambda)^2 C_1 - 2[2(1+\lambda) C_1 - (2+\lambda)] + \frac{10}{3} (C_1 - 1) - 2[2\lambda(1+\lambda) C_1 - (1+2\lambda)] \\
& + 4[2\lambda C_1 - (1+\lambda)] + \frac{26}{5} \lambda (\lambda C_1 - 1) = I_2
\end{aligned}$$

$$\frac{1+\lambda}{4} - \frac{1}{6} - \frac{\lambda}{8} = \frac{1}{12} + \frac{\lambda}{8} = I_3$$

$$\rho^2 \frac{E(\frac{1}{7})}{4} \left\{ \overbrace{A_3 C_1^3 + A_2 C_1^2 + A_1 C_1 + A_0}^{I_1} \right\} + \frac{E(\frac{1}{7})}{12} \frac{1}{\rho^2} \left\{ \overbrace{B_1 C_1 + B_0}^{I_2} \right\} + \frac{1}{4} \left\{ \frac{1}{3} + \frac{1}{2} \right\} = 0 \quad (2.5)$$

$$\rho^2 \{ A_3 C_1^3 + A_2 C_1^2 + A_1 C_1 + A_0 \} + \frac{(\frac{1}{7})}{3} \frac{1}{\rho^2} \{ B_1 C_1 + B_0 \} + (\frac{20}{E}) \left(\frac{1}{3} + \frac{1}{2} \right) = 0.$$

$$\rho^2 = \frac{(\frac{1}{7})}{\sqrt{3}} (B_1 C_1 + B_0)^{\frac{1}{2}} (A_3 C_1^3 + A_2 C_1^2 + A_1 C_1 + A_0)^{-\frac{1}{2}}$$

$$\frac{1}{\sqrt{3}} \left[(B_1 C_1 + B_0) (A_3 C_1^3 + A_2 C_1^2 + A_1 C_1 + A_0) \right]^{\frac{1}{2}} + \phi \left(\frac{1}{3} + \frac{1}{2} \right) = 0.$$

$$B_1 (A_3 C_1^3 + A_2 C_1^2 + A_1 C_1 + A_0) + (B_1 C_1 + B_0) (3A_3 C_1^2 + 2A_2 C_1 + A_1) = 0$$

$$\text{or } (4A_3 B_1) C_1^3 + 3(B_1 A_2 + B_0 A_3) C_1^2 + 2(A_1 B_1 + A_2 B_0) C_1 + (A_1 B_0 + B_1 A_0) = 0$$

$$A_3 = \frac{2}{3} (1+\lambda)^4 - 2(1+\lambda)^3 + \frac{6}{5} (1+\lambda)^2 (3-2\lambda-2\lambda^2) + \frac{6}{3} (1+\lambda) (2+\lambda) \\ + \frac{2}{7} [6\lambda^2 (1+\lambda)^2 - 12\lambda (1+\lambda) + 1] + \lambda (1-3\lambda-3\lambda^2) + \frac{6\lambda^2}{7} [3-2\lambda-4\lambda^2] \\ + \frac{6}{5} \lambda^3 + \frac{2\lambda^4}{11} \quad \text{OK } \checkmark \checkmark$$

$$A_2 = \left\{ -\frac{3}{2} (1+\lambda)^3 + \frac{6}{5} (1+\lambda)^2 (2-\lambda) + 3\lambda (1+\lambda) + \frac{6}{7} (3\lambda^2 - 6\lambda - 2) \right\}$$

should be 0

$$+ \frac{6}{7} \left[(3\lambda+1) - 3\lambda (1+\lambda) (2+\lambda) \right] + 2\lambda (1+\lambda) (1-\lambda) + \frac{2}{5} \lambda^2 (3+\lambda) + \frac{6}{11} \lambda^3 \}$$

OK

$$A_1 = \left\{ -\frac{3}{3} (1+\lambda)^2 + (1+\lambda) + \frac{6}{5} (5\lambda^2 + 8\lambda + 2) + \frac{2}{3} \lambda (3\lambda^2 + 2) + \frac{2}{7} (2+\lambda) \right. \\ \left. + (1+\lambda)(1-\lambda) + \frac{2}{3} (1+2\lambda-\lambda^2) + \frac{6}{5} \lambda (1+\lambda) + \frac{6\lambda^2}{11} \right\}$$

I think this should be $-3\lambda(1+\lambda)$ (O.K.)

I think this should be $-\frac{6}{7} (2\lambda^2 + 6\lambda + 3)$

$$A_0 = -\frac{1}{2}(1+\lambda) - \frac{1}{5}\lambda + \frac{2}{3}(1+\lambda) + \frac{\frac{4}{7}(1+2\lambda)}{\text{OK}} + \frac{1}{2} \left(-\frac{4}{7}(1+\lambda) \right) - \frac{1}{5}(1+\lambda) \quad 246)$$

$\left(+\frac{2}{11}\lambda \right)$ should be \ominus

$$B_1 = \frac{2(1+\lambda)^2 - 4(1+\lambda) + \frac{10}{3}}{\text{OK}} - \frac{4\lambda(1+\lambda)}{\text{OK}} + \frac{8\lambda}{\text{OK}} + \frac{26}{5}\lambda^2 \quad \text{OK}$$

$$B_0 = 2(2+\lambda) - \frac{10}{3} + 2(1+2\lambda) - 4(1+\lambda) - \frac{26}{5}\lambda \quad \text{OK}$$

(214)

 A_3

① λ	② $1+\lambda$	③ $(1+\lambda)^2$	④ $(1+\lambda)^3$	⑤ $(1+\lambda)^4$	⑥ $\frac{2}{3} \text{ ⑤}$	⑦ -2 ④	⑧ $\lambda(1+\lambda)$	⑨ 2 ⑧
-0.5	0.5	0.25	0.125	0.0625	0.04167	-0.250	-0.250	-0.500
-0.4	0.6	0.36	0.216	0.1296	0.08640	-0.432	-0.240	-0.480
-0.3	0.7	0.49	0.343	0.2401	0.16007	-0.686	-0.210	-0.420
-0.2	0.8	0.64	0.512	0.4096	0.27307	-1.024	-0.160	-0.320
-0.1	0.9	0.81	0.729	0.6561	0.43740	-1.458	-0.090	-0.180
0	1.0	1.00	1.000	1.0000	0.66667	-2.000	0	0
+0.1	1.1	1.21	1.331	1.4641	0.97607	-2.662	+0.110	+0.220
+0.2	1.2	1.44	1.728	2.0736	1.38240	-3.456	+0.240	+0.480
+0.3	1.3	1.69	2.197	2.8561	1.90407	-4.394	+0.390	+0.780
+0.4	1.4	1.96	2.744	3.8416	2.56107	-5.488	+0.560	+1.120
+0.5	1.5	2.25	3.375	5.0625	3.36834	-6.750	+0.750	+1.500
	⑩	⑪	⑫	⑬	⑭	⑮	⑯	⑰
λ	$3 - \text{⑨}$	$\frac{4}{3} \text{ ⑩}$	$\text{⑪} \times \text{③}$	$3 \times \text{⑧}$	$\text{⑬} - 1$	$\frac{4}{3} \times \text{⑭}$	$\text{⑯} \times \text{②}$	λ^2
-0.5	3.500	2.600	0.7000	-0.750	-1.750	-2.33333	-1.16667	0.25
-0.4	3.480	2.784	1.00224	-0.720	-1.720	-2.29333	-1.37599	0.16
-0.3	3.420	2.736	1.34064	-0.630	-1.630	-2.17333	-1.52133	0.09
-0.2	3.320	2.656	1.69984	-0.480	-1.480	-1.97333	-1.57866	0.04
-0.1	3.180	2.544	2.06064	-0.270	-1.270	-1.69333	-1.52399	0.01
0	3.000	2.400	2.40000	0	-1.000	-1.33333	-1.33333	0
+0.1	2.780	2.224	2.69104	+0.330	-0.670	-0.89333	-0.91266	0.01
+0.2	2.520	2.016	2.90304	+0.720	-0.280	-0.37333	-0.44799	0.04
+0.3	2.220	1.776	3.00144	+1.170	+0.170	+0.22667	+0.29467	0.09
+0.4	1.880	1.504	2.94784	+1.680	+0.680	+0.90667	+1.26934	0.16
+0.5	1.500	1.200	2.70000	+2.250	+1.250	+1.66667	+2.50000	0.25

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 A_4

	(18)	(19)	(20)	(21)	(22)	(23)	(24)
λ	$\beta - \lambda$	$2 \times (13)$	$(18) \times (19)$	$(20) + 1$	$\frac{2}{3} (21)$	$-\lambda \cdot (14)$	$6\lambda^2$
-0.5	-2.250	-1.500	+3.3750	4.3750	1.25000	-0.175	1.000
-0.4	-2.240	-1.440	+3.2256	4.2256	1.20731	-0.688	0.640
-0.3	-2.210	-1.260	+2.7846	3.7846	1.05131	-0.489	0.360
-0.2	-2.160	-0.960	+2.0736	3.0736	0.87817	-0.296	0.160
-0.1	-2.090	-0.540	+1.1286	2.1286	0.60817	-0.127	0.040
0	-2.000	0	0	1.0000	0.28571	0	0
+0.1	-1.890	+0.660	-1.2474	-0.2474	-0.07069	+0.067	0.040
+0.2	-1.760	+1.440	-2.5344	-1.5344	-0.43540	+0.056	0.160
+0.3	-1.610	+2.340	-3.7674	-2.7674	-0.79068	-0.051	0.360
+0.4	-1.440	+3.360	-4.8384	-3.8384	-1.09668	-0.272	0.640
+0.5	-1.250	+4.500	-5.6250	-4.6250	-1.32143	-0.625	1.000
	(25)	(26)	(27)	(28)	(29)	(30)	(31)
λ	2λ	$(25) + (24)$	$3 - (24)$	$\frac{4}{3} \times (23)$	$(13) \times 28$	$\frac{4}{3} \lambda^3$	λ^4
-0.5	-1.0	0	3.000	1.33333	0.33333	-0.1000	0.0625
-0.4	-0.8	-0.16	3.160	1.40444	0.22471	-0.0512	0.0256
-0.3	-0.6	-0.24	3.240	1.44000	0.1296	-0.0216	0.0081
-0.2	-0.4	-0.24	3.240	1.44000	0.05760	-0.0064	0.0016
-0.1	-0.2	-0.16	3.160	1.40444	0.01404	-0.0008	0.0001
0	0	0	3.000	1.33333	0	0	0
+0.1	+0.2	+0.24	2.760	1.22666	0.012267	+0.0008	0.0001
+0.2	+0.4	+0.56	2.440	1.08444	0.043378	+0.0064	0.0016
+0.3	+0.6	+0.96	2.040	0.90667	0.081600	+0.0216	0.0081
+0.4	+0.8	+1.44	1.560	0.69333	0.110933	+0.0512	0.0256
+0.5	+1.0	+2.00	1.000	0.44444	0.111111	+0.1000	0.0625

A_3

(32)		(33)				
λ	$-\frac{2}{\pi} \lambda^2$	(A_3)				
-0.5	0.01136	-0.05531				
-0.4	0.00465	-0.02188				
-0.3	0.00147	-0.00484				
-0.2	0.00029	+0.00391				
-0.1	0.00002	+0.01048				
0	0	+0.01905				
+0.1	0.00002	+0.03185				
+0.2	0.00029	+0.04912				
+0.3	0.00147	+0.06917				
+0.4	0.00465	+0.08835				
+0.5	0.01136	+0.09438				

$$\begin{aligned}
 A_2 &= - \left\{ -15(\lambda^3 + 3\lambda^2 + 3\lambda + 1) + 1.2(-\lambda^3 + 3\lambda + 2) + 3(\lambda^3 + 3\lambda^2 + 2\lambda) \right. \\
 &\quad + 0.85714286(3\lambda^3 - 6\lambda - 2) + 0.75(-3\lambda^3 - 9\lambda^2 - 3\lambda + 1) \\
 &\quad \left. + 2(-\lambda^3 + \lambda) + 0.6(\lambda^3 + 3\lambda) + 0.54545454\lambda^3 \right\} \\
 &= \left\{ (1.5 + 1.2 - 3 - 2.5714286 + 2.25 + 2 - 0.6 - 0.545454) \lambda^3 \right. \\
 &\quad + (4.5 - 1 + 6.75) \lambda^2 \\
 &\quad + (4.5 - 3.6 - 6 + 5.1428571 + 2.25 - 2 - 1.8) \lambda \\
 &\quad \left. + (1.5 - 2.4 + 1.7142857 - 0.25) \right\}
 \end{aligned}$$

$$\boxed{A_2 = 0.23312 \lambda^3 + 2.25000 \lambda^2 - 1.50714 \lambda + 0.06429}$$

$$\begin{aligned}
 A_1 &= \left\{ -0.666667(\lambda^2 + 2\lambda + 1) + (\lambda + 1) + 0.40(5\lambda^2 + 2\lambda + 2) + 0.666667(3\lambda^2 + 2\lambda) \right. \\
 &\quad - \frac{2}{7}(2\lambda^2 + 24\lambda + 15) - 3(\lambda^2 + \lambda) + \frac{1}{3}(-\lambda^2 + 2\lambda + 1) + 1.2(\lambda^2 + \lambda) \\
 &\quad \left. + 0.54545454\lambda^2 \right\}
 \end{aligned}$$

$$\begin{aligned}
 &= \left\{ (0.666667 + \cancel{2} - 3 + 1.2 + 0.545454) \lambda^2 \right. \\
 &\quad + (+1.33333 + 1 + 3.2 - 6.85714 - 3 + 1.2) \lambda \\
 &\quad \left. + (1 + 0.8 - 4.285714) \right\}
 \end{aligned}$$

$$\boxed{A_1 = -0.58718 \lambda^2 - 3.12381 \lambda - 2.41571}$$

$$A_0 = (\cancel{-0.5} - \cancel{0.4} + 0.666667 + \cancel{1.142857} + \cancel{0.5} - 0.444444 - \cancel{0.6} + 0.111111) \lambda \quad 22/)$$

$$(-0.5 + 0.666667 + 0.571428 - 0.444444 - 0.2)$$

$$A_0 = 0.25398 \lambda + 0.09366$$

$$B_1 = (2 - 4 + 5.2) \lambda^2 + (4 - 4 - 4 + 8) \lambda + (2 - 4 + 3.3333)$$

$$B_1 = 3.2 \lambda^2 + 4 \lambda + 1.33333$$

$$B_0 = (2 + 4 - 4 - 5.2) \lambda + (4 - 3.3333 + 2 - 4)$$

$$B_0 = -3.2 \lambda - 1.33333$$

222)

①	②	③	④	⑤	⑥	⑦
λ	λ^2	λ^3	$0.23312\lambda^2$	$2.25000\lambda^2$	-1.50714λ	A_2
-0.5	0.25	-0.125	-0.02914	0.56250	+0.25357	1.35122
-0.4	0.16	-0.064	-0.01492	0.36000	+0.60286	1.01223
-0.3	0.09	-0.027	-0.00629	0.20250	+0.45214	0.71264
-0.2	0.04	-0.008	-0.00186	0.09000	+0.30143	0.45386
-0.1	0.01	-0.001	-0.00023	0.02250	+0.15071	0.23727
0	0	0	0	0	0	0.06429
+0.1	0.01	+0.001	+0.00023	0.02250	-0.15071	-0.06369
+0.2	0.04	+0.008	+0.00186	0.09000	-0.30143	-0.16528
+0.3	0.09	+0.027	+0.00629	0.20250	-0.45214	-0.17906
+0.4	0.16	+0.064	+0.01492	0.36000	-0.60286	-0.16365
+0.5	0.25	+0.125	+0.02914	0.56250	-0.25357	-0.09764
⑧	⑨	⑩	⑪	⑫		
λ	$-0.56282\lambda^2$	-3.12381λ	A_1	0.25398λ	A_0	
-0.5	-0.14697	+1.56191	-1.07077	-0.12699	-0.03333	
-0.4	-0.09406	+1.24952	-1.33025	-0.10159	-0.00793	
-0.3	-0.05291	+0.93714	-1.60148	-0.07619	+0.01747	
-0.2	-0.02352	+0.62476	-1.88447	-0.05080	+0.04246	
-0.1	-0.00588	+0.31238	-2.17921	-0.02540	+0.06826	
0	0	0	-2.48571	0	+0.09366	
+0.1	-0.00588	-0.31238	-2.80597	+0.02540	+0.11906	
+0.2	-0.02352	-0.62476	-3.13399	+0.05080	+0.14446	
+0.3	-0.05291	-0.93714	-3.47576	+0.07619	+0.16985	
+0.4	-0.09406	-1.24952	-3.82929	+0.10159	+0.19525	
+0.5	-0.14697	-1.56191	-4.19459	+0.12699	+0.22065	

223)

 K_1

①	②	③	④	⑤	⑥	⑦
λ	$3.2\lambda^2$	4λ	$[B_1]$	$[B_0]$	$A_3 B_1$	$4A_3 B_1$
-0.5	0.8	-2.0	+0.13333	+0.26667	-0.007374	-0.02950
-0.4	0.512	-1.6	+0.24533	-0.05333	-0.005368	-0.02147
-0.3	0.288	-1.2	+0.42133	-0.37333	-0.002039	-0.00816
-0.2	0.128	-0.8	+0.66133	-0.69333	+0.002586	+0.01034
-0.1	0.032	-0.4	+0.96533	-1.01333	+0.010117	+0.04047
0	0	0	+1.33333	-1.33333	+0.025400	+0.10160
+0.1	0.032	+0.4	+1.76533	-1.65333	+0.056226	+0.22490
+0.2	0.128	+0.8	+2.26133	-1.97333	+0.111077	+0.44431
+0.3	0.288	+1.2	+2.82133	-2.29333	+0.195151	+0.78060
+0.4	0.512	+1.6	+3.44533	-2.61333	+0.304395	+1.21758
+0.5	0.800	+2.0	+4.13333	-2.93333	+0.390104	+1.56042
	⑧	⑨	⑩	⑪	⑫	⑬
λ	$A_2 B_1$	$A_3 B_0$	⑥+⑨	3⑩ K_2	$A_1 B_1$	$A_2 B_0$
-0.5	+0.16016	-0.01475	+0.16541	+0.49623	-0.14277	+0.36033
-0.4	+0.24833	+0.00117	+0.24950	+0.74850	-0.32635	-0.05398
-0.3	+0.30026	+0.00181	+0.30207	+0.90621	-0.67475	-0.26605
-0.2	+0.30015	-0.00271	+0.29744	+0.89232	-1.24626	-0.31467
-0.1	+0.22904	-0.01062	+0.21842	+0.65526	-2.10366	-0.24543
0	+0.08572	-0.02540	+0.06032	+0.18096	-3.31427	-0.08572
+0.1	-0.11243	-0.05266	-0.16509	-0.49527	-4.94993	+0.10530
+0.2	-0.32453	-0.09693	-0.42546	-1.24138	-7.08699	+0.21669
+0.3	-0.50519	-0.15863	-0.66382	-1.99146	-9.80627	+0.41064
+0.4	-0.56383	-0.23089	-0.79472	-2.38416	-13.19317	+0.42767
+0.5	-0.40358	-0.24685	-0.68043	-2.04129	-17.33762	+0.25641

224)

K_3				K_4		
	(14)	(15)	(16)	(17)	(18)	
λ	(13) + (13)	2 * (14)	A, S,	A, B,	(16) + (17)	
-0.5	+0.21756	+0.4351	-0.004444	-0.285542	-0.28998	
-0.4	-0.27237	-0.5448	-0.001945	+0.070942	0.06899	
-0.3	-0.94080	-1.8816	+0.006097	+0.597121	0.60398	
-0.2	-1.56093	-3.1218	+0.028345	+1.30656	1.3349	
-0.1	-2.34409	-4.6882	+0.065873	+2.20826	2.2742	
0	-3.39999	-6.7999	+0.124180	+3.31427	3.4392	
+0.1	-4.84463	-9.6893	+0.210180	+4.63589	4.8461	
+0.2	-6.80030	-13.6006	+0.326672	+6.18440	6.5111	
+0.3	-9.37563	-18.7913	+0.47703	+7.92106	8.4503	
+0.4	-12.71550	-25.5310	+0.672701	+10.0072	10.6799	
+0.5	-17.05121	-34.1024	+0.912017	+12.3041	13.2161	
	(19)	(20)	(21)	(22)	(23)	(24)
λ				$\xi = -\frac{\omega_1}{3}$	(19) * (22)	$\rho = (20) + (23)$
-0.5	-16.821	-14.749	+9.8298	+5.6070	-94.315	-109.064
-0.4	-34.863	+25.375	-3.2133	+11.621	-405.143	-379.768
-0.3	-111.06	+230.59	-74.017	+37.020	-4111.44	-3880.85
-0.2	+86.298	-301.91	+129.10	-28.766	-2482.45	-2784.36
-0.1	+16.191	-115.84	+56.195	-5.3970	-87.383	-203.223
0	+1.7811	-66.928	+33.850	-0.59370	-1.0574	-67.915
+0.1	-2.2022	-43.013	+21.548	+0.73407	-1.6166	-44.700
+0.2	-2.8727	-30.611	+14.654	+0.95757	-2.7508	-33.362
+0.3	-2.5512	-24.073	+10.125	+0.85040	-2.1695	-26.243
+0.4	-1.9581	-20.969	+8.7214	+0.65270	-1.2781	-22.247
+0.5	-1.3022	-21.855	+8.4696	+0.43607	-0.57047	-22.425

$$C_1^3 + \alpha_1 C_1^2 + \alpha_2 C_1 + \alpha_3 = 0$$

Put $C_1 = x + \xi$

$$x^3 + 3x^2\xi + 3x\xi^2 + \xi^3 + \alpha_1(x^2 + 2x\xi + \xi^2) + \alpha_2(x + \xi) + \alpha_3 = 0$$

$$x^3 + \underbrace{\{3\xi + \alpha_1\}}_{\substack{\parallel \\ 0}} x^2 + \underbrace{\{3\xi^2 + 2\alpha_1\xi + \alpha_2\}}_p x + \underbrace{\{\xi^3 + \alpha_1\xi^2 + \alpha_2\xi + \alpha_3\}}_q = 0.$$

$$\xi = -\frac{\alpha_1}{3}$$

$$p = 3\xi^2 + 2\alpha_1\xi + \alpha_2 = \frac{\alpha_1^2}{3} - \frac{2}{3}\alpha_1^2 + \alpha_2 = \alpha_2 - \frac{\alpha_1^2}{3}$$

$$q = \xi^3 + \alpha_1\xi^2 + \alpha_2\xi + \alpha_3 = -\frac{\alpha_1^3}{27} + \frac{\alpha_1^3}{9} - \frac{\alpha_1\alpha_2}{3} + \alpha_3$$

$p =$	$\alpha_2 - \frac{\alpha_1^2}{3}$
$q =$	$\frac{2}{27}\alpha_1^3 - \frac{\alpha_1\alpha_2}{3} + \alpha_3$

$$q = \frac{\alpha_1}{3} \left[\frac{2}{9}\alpha_1^2 - \alpha_2 \right] + \alpha_3$$

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	(23)	(24)	(25)	(26)	(27)	(28)
λ	$-u, (23)$	$\frac{2}{3} u, (24)$	$(25) \times (20)$	$(26) + (27) + (28)$	$-f/3$	$r^2 = (29)^2$
-0.5	-1586.47	-352.549	-82698	-425.417	36.355	6.0495
-0.4	-14124.5	-3138.78	+299883	-2847.11	126.589	11.251
-0.3	-456616.5	-101470.3	+853644	93007.9	1293.62	35.967
-0.2	+214230.5	+47606.77	+868474	+564206	928.120	10.465
-0.1	+141482	+314.404	+625.188	+995.787	67.7410	8.2305
0	+188334	+041852	+397352	+24.004	22.662	4.7605
0.1	-3.86008	-0.9113	-316259	-10869	14900	3.8601
0.2	-7.90222	-1.75605	-29.3122	-16.414	11.121	3.3348
0.3	-5.53483	-1.22996	-20.4717	-10.877	8.7477	2.9576
0.4	-2.50265	-0.55614	-13.6865	-5.472	7.4157	2.7232
0.5	-0.74629	-0.16584	-9.5303	-1.2265	7.4750	2.7341
	(31)	(32)	(33)	(34)	(35)	(36)
λ	r	$-9/2 \div (31)$	θ	ρ_1	ρ_2	ρ_3
-0.5	219.20	+0.97039	13°58.7'	4°39.6'	124°39.6'	244°39.6'
-0.4	1424.3	+0.99948	1°51.0'	37'	120°37'	240°37'
-0.3	46527	+0.99951	1°48.0'	36'	120°36'	240°36'
-0.2	28275	-0.99771	183°52.5'	61°17.5'	181°17.5'	301°17.5'
-0.1	557.54	-0.89302	206°44.7'	68°52.9'	188°52.9'	308°52.9'
0	107.88	-0.31518	102°22.3'	34°2.4'	154°2.4'	274°2.4'
+0.1	57515	+0.894488	84°38.7'	28°11.6'	148°11.6'	268°11.6'
+0.2	37.086	+0.22130	77°12.9'	25°44.3'	145°44.3'	265°44.3'
+0.3	25.873	+0.21020	77°52.0'	25°57.3'	145°57.3'	265°57.3'
+0.4	20.194	+0.13549	82°12.8'	27°24.3'	147°24.3'	267°24.3'
+0.5	20.437	+0.030007	88°16.8'	29°25.6'	149°25.6'	269°25.6'

227)

	(37)	(38)	(39)	(40)	(41)	(42)
λ	$\cos \beta_1$	$\cos \beta_2$	$\cos \beta_3$	λ_1	λ_2	λ_3
-0.5	0.99669	-0.56870	-0.42799	-6.8580	-5.1611	+12.019
-0.4	0.99994	-0.50929	-0.49065	-11.460	-11.041	+22.501
-0.3	0.99995	-0.50904	-0.49090	-36.617	-35.312	+71.930
-0.2	0.48035	-0.99975	+0.51940	-60.915	+29.268	+31.647
-0.1	0.36030	-0.98801	+0.62772	-16.264	+5.9309	+10.3329
0	0.82783	-0.90100	+0.07481	-8.5784	+0.71227	+7.8818
0.1	0.88136	-0.84983	-0.03153	-6.5609	-0.24342	+6.8043
0.2	0.90078	-0.82648	-0.07431	-5.5123	-0.49562	+6.0078
0.3	0.89914	-0.82844	-0.07054	-4.9004	-0.41226	+5.3186
0.4	0.88778	-0.84250	-0.04527	-4.5886	-0.24656	+4.8352
0.5	0.87099	-0.86098	-0.01011	-4.7080	-0.05528	+4.7627
	(43)	(44)	(45)			
λ	$C_1^{(1)}$	$C_1^{(2)}$	$C_1^{(3)}$			
-0.5	-1.2510	+0.4459	+17.626			
-0.4	+0.161	+0.580	+34.122			
-0.3	+0.403	+1.708	+108.95			
-0.2	-89.681	+0.502	+2.881			
-0.1	-21.661	+0.5339	+4.9359			
0	-9.1721	+0.11857	+7.2881			
0.1	-5.8268	+0.49065	+7.5384			
0.2	-4.5547	+0.46195	+6.9654			
0.3	-4.0500	+0.43314	+6.1690			
0.4	-3.9359	+0.40614	+5.4879			
0.5	-4.2719	+0.38079	+5.1988			

	①	②	③	④	⑤
λ	$[C_1^{(1)}]^2$	$[C_1^{(2)}]^3$	$[C_1^{(3)}]^2$	$[C_1^{(4)}]^3$	$[C_1^{(5)}]^3$
-0.5	1.5650	+0.19883	+310.68	-1.9578	+0.088658
-0.4	+0.02592	+0.3364	+1164.31	+0.004173	+0.19511
-0.3	+0.16241	+2.9173	+11870.1	+0.065451	+9.9827
-0.2	+8042.7	+0.2520	+8.3002	-721.276	+0.12650
-0.1	+469.20	+0.28505	+24.3631	-10163.3	+0.152188
0	+84.1274	+0.014059	+53.116	-771.625	+0.0016620
+0.1	+33.951	+0.24069	+56.827	-197829	+0.11809
+0.2	+20.745	+0.21340	+48.517	-94.487	+0.09858
+0.3	+16.403	+0.18761	+38.057	-66.432	+0.08126
+0.4	+15.491	+0.16495	+30.117	-60.971	+0.06699
+0.5	+18.249	+0.14500	+27.028	-77.958	+0.05521

	⑥				
λ	$[C_1^{(6)}]^3$				
-0.5	5475.98				
-0.4	39728.6				
-0.3	1293247				
-0.2	+23.913				
-0.1	+120.254				
0	+387.12				
+0.1	+428.38				
+0.2	+337.94				
+0.3	+234.77				
+0.4	+165.28				
+0.5	+140.51				

$$\textcircled{5} = B_0$$

229)

	①	②	③	④	⑤	⑥
λ	$B, C, {}^{(1)}$	$\textcircled{5} + \textcircled{1}$	$B, C, {}^{(2)}$	$\textcircled{5} + \textcircled{4}$	$B, C, {}^{(3)}$	$\textcircled{5} + \textcircled{5}$
-0.5	-0.16680	+0.09987	+0.05945	+0.32612	+2.3500	2.6167
-0.4	+0.03950	-0.01383	+0.14229	+0.08896	+8.3712	8.3179
-0.3	+0.16980	-0.20353	+0.71963	+0.34630	+45.904	45.531
-0.2	-59.309	-60.002	+0.33199	-0.36134	+1.9053	1.2120
-0.1	-20.910	-21.923	+0.51539	-0.49794	+4.7648	3.7515
0	-12.229	-13.562	+0.15809	-1.17524	+9.7174	8.3841
+0.1	-10.286	-11.939	+0.86616	-0.78717	+13.3077	11.6544
+0.2	-10.300	-12.273	+1.04462	-0.92871	+15.7511	13.7228
+0.3	-11.426	-13.719	+1.2220	-1.0711	+17.405	15.112
+0.4	-13.560	-16.173	+1.3993	-1.2140	+18.908	16.295
+0.5	-17.657	-20.590	+1.5739	-1.3594	+21.488	18.555

230)

$$(B_1 C_1 + B_0)(A_3 C_1^3 + A_2 C_1^2 + A_1 C_1 + A_0)$$

$$= - \frac{(B_1 C_1 + B_0)^2 (3A_3 C_1^2 + 2A_2 C_1 + A_1)}{B_1}$$

$$\phi = \frac{1}{(\frac{1}{3} + \frac{1}{2})} \left\{ - (B_1 C_1 + B_0) \right\} \left\{ \frac{-(3A_3 C_1^2 + 2A_2 C_1 + A_1)}{3B_1} \right\}^{\frac{1}{2}}$$

231)

	①	②	③	④
λ	$3A_3 [C_1^{(1)}]^2$	$2A_2 C_1^{(1)}$	$A_1 + ① + ②$	$-③/3B_1$
-0.5	-0.25968	-33.808	-4.7113	11.778
-0.4	-0.001701	+0.3259	-1.0061	1.3670
-0.3	-0.002358	+0.57439	-1.0295	0.81448
-0.2	+94.341	-81.405	+11.052	
-0.1	+14.752	-10.279	+2.294	
0	+4.8079	-1.1293	+1.1429	
0.1	+3.2440	+0.74221	+1.1822	
0.2	+3.0570	+1.3234	+1.2464	
0.3	+3.4038	+1.4504	+1.3784	
0.4	+4.1062	+1.2882	+1.5651	
0.5	+5.1670	+0.8342	+1.8066	
	⑤	⑥	⑦	⑧
λ	$3A_3 [C_1^{(1)}]^2$	$2A_2 C_1^{(1)}$	$A_1 + ⑤ + ⑥$	$-⑦/3B_1$
-0.5	-0.03299	+1.2050	+0.1012	
-0.4	-0.02208	+1.1742	-0.1782	0.2421
-0.3	-0.042359	+2.4344	+0.7905	
-0.2	+0.002955	+0.4557	-1.4258	0.7186
-0.1	+0.008962	+0.25336	-1.9169	0.6619
0	+0.0008035	+0.01525	-2.4697	0.6174
0.1	+0.022998	-0.06250	-2.8435	0.5369
0.2	+0.031447	-0.13422	-3.2368	0.4771
0.3	+0.038931	-0.15512	-3.5920	0.4244
0.4	+0.043720	-0.13293	-3.9185	0.3791
0.5	+0.041055	-0.07436	-4.2279	0.3410

2

3

232)

	⑧	⑩	⑪	⑫
λ	$3A_3[C_1^{(3)}]^2$	$2A_2 C_1^{(3)}$	$A_1 + ⑧ + ⑩$	$- ⑪ / 3B_1$
-0.5	- 51.551	+ 47.633	- 4.989	12.473
-0.4	- 26.425	+ 69.079	- 8.676	11.788
-0.3	- 12.35	+ 270.09		
-0.2	+ 0.09736	+ 2.6151		
-0.1	+ 0.76598	+ 2.3423		
0	+ 3.0356	+ 0.93710	+	
0.1	+ 5.4298	- 0.96024		
0.2	+ 7.1495	- 7.0239		
0.3	+ 7.8972	- 2.2092		
0.4	+ 7.9825	- 1.7962		
0.5	+ 7.6527	- 1.0152		
	⑬	⑭	⑮	⑯
λ	$6/2 + 3\lambda$	$\phi^{(1)}$	$\phi^{(2)}$	$\phi^{(3)}$
-0.5	12	- 4.11		
-0.4	7.5	+ 0.1213		
-0.3	5.45454	+ 1.000		
-0.2	4.28571			
-0.1	3.5294			
0	3			
0.1	2.60870			
0.2				
0.3				
0.4				
0.5				

$$\begin{aligned}
& \frac{2}{3}(1+\lambda)^2[(1+\lambda)^2 C_1^3 - C_1] - \frac{1}{2}(1+\lambda)[4(1+\lambda)^2 C_1^3 - 3(1+\lambda)^2 C_1^2 - 2C_1 + 1] \quad 233) \\
& + \frac{2}{5} \left\{ 2(1+\lambda)^2[3 - 2\lambda(1+\lambda)]C_1^3 - 9(1+\lambda)^2 C_1^2 + [3(1+\lambda)^2 + \lambda(1+\lambda) - 1]C_1 + 1 \right\} \\
& + \frac{1}{3} \left\{ 4(1+\lambda)[3\lambda(1+\lambda) - 1]C_1^3 + 9(1+\lambda)[1 - \lambda(1+\lambda)]C_1^2 - 2[\lambda + 3(1+\lambda)]C_1 \right. \\
& \quad \left. + (1 + 2\lambda) \right\} \\
& + \frac{2}{7} \left\{ [6\lambda^2(1+\lambda)^2 - 12\lambda(1+\lambda) + 1]C_1^3 + 3[6\lambda(1+\lambda) - 1]C_1^2 \right. \\
& \quad \left. - [\lambda^2 + 6\lambda(1+\lambda) - 3]C_1 - 1 \right\} \\
& + \frac{1}{4} \left\{ 4\lambda[1 - 3\lambda(1+\lambda)]C_1^3 + 9\lambda[\lambda(1+\lambda) - 1]C_1^2 + 6\lambda C_1 - \lambda \right\} \\
& + \frac{1}{9} \left\{ 4\lambda^2[3 - 2\lambda(1+\lambda)]C_1^3 - 18\lambda^2 C_1^2 + 6\lambda^2 C_1 \right\} \\
& + \frac{1}{5} \left\{ 4\lambda^3 C_1^3 - 3\lambda^3 C_1^2 \right\} + \frac{2}{11} \lambda^4 C_1^3 = \underline{7}. \quad \text{multiplied by } \rho^6
\end{aligned}$$

$$\begin{aligned}
& \frac{2(1+\lambda)[C_1(1+\lambda) - 1]}{2(1+\lambda)C_1 - 2[2(1+\lambda)C_1 - (2+\lambda)] + \frac{10}{3}(C_1 - 1) - 2\lambda[2(1+\lambda)C_1 - 1]} \\
& + 4\lambda[2C_1 - 1] + \frac{26}{5}\lambda^2 C_1 = \underline{1}
\end{aligned}$$

$$\frac{(1+\lambda)}{4} - \frac{1}{6} - \frac{\lambda}{8} = \frac{1}{12} + \frac{\lambda}{8} \quad \underline{\underline{1}}$$

$$-4(C_1 - 1)$$

$$A_3 = \frac{2}{3}(1+\lambda)^4 - 2(1+\lambda)^3 + \frac{4}{5}(1+\lambda)^2[3-2\lambda(1+\lambda)] + \frac{4}{3}(1+\lambda)[3\lambda(1+\lambda)-1] \quad 234)$$

$$+ \frac{2}{7}[6\lambda^2(1+\lambda)^2 - 12\lambda(1+\lambda)+1] + \lambda[1-3\lambda(1+\lambda)] + \frac{4}{7}\lambda^2[3-2\lambda(1+\lambda)]$$

$$+ \frac{4}{5}\lambda^3 + \frac{2}{11}\lambda^4$$

$$A_2 = \frac{3}{2}(1+\lambda)^3 - \frac{18}{5}(1+\lambda)^2 + 3(1+\lambda)[1-\lambda(1+\lambda)] + \frac{6}{7}[6\lambda(1+\lambda)-1]$$

$$+ \frac{9}{4}\lambda[\lambda(1+\lambda)-1] - 2\lambda^2 - \frac{3}{5}\lambda^3$$

$$A_1 = -\frac{2}{3}(1+\lambda)^2 + (1+\lambda) + \frac{2}{5}[3(1+\lambda)^2 + \lambda(1+\lambda)-1] - \frac{2}{3}[\lambda + 3(1+\lambda)]$$

$$- \frac{2}{7}[\lambda^2 + 6\lambda(1+\lambda)-3] + \frac{3}{2}\lambda + \frac{2}{3}\lambda^2$$

$$A_0 = -\frac{1}{2}(1+\lambda) + \frac{2}{5} + \frac{1+2\lambda}{3} - \frac{2}{7} - \frac{1}{4} = -0.63333\lambda - 0.52381$$

$$B_1 = 2(1+\lambda)^2 - 4(1+\lambda) + \frac{10}{3} - 4\lambda(1+\lambda) + 8\lambda + \frac{26}{5}\lambda^2$$

$$\boxed{B_0 = \frac{2(2+\lambda) - \frac{10}{3} - 2\lambda}{-2(1+\lambda)} = -2\lambda - 1.33333}$$

$$B_1 = 2(\lambda^2 + 2\lambda + 1) - 4(\lambda + 1) + \frac{10}{3} - 4(\lambda^2 + \lambda) + 8\lambda + \frac{26}{5}\lambda^2$$

$$= (2 - 4 + \frac{26}{5})\lambda^2 + (4 - 4 - 4 + 8)\lambda + (2 - 4 + \frac{10}{3})$$

$$\boxed{B_1 = 3.2\lambda^2 + 4\lambda + 1.33333}$$

$$A_3 = \frac{2}{3}(\lambda^4 + 4\lambda^3 + 6\lambda^2 + 4\lambda + 1) - 2(\lambda^3 + 3\lambda^2 + 3\lambda + 1) + \frac{4}{5}(3 + 4\lambda - 3\lambda^2 - 6\lambda^3 - 2\lambda^4)^{235}) \\ + \frac{4}{3}(3\lambda^3 + 6\lambda^2 + 2\lambda - 1) + \frac{2}{7}(6\lambda^4 + 12\lambda^3 - 6\lambda^2 - 12\lambda + 1) + (\lambda - 3\lambda^2 - 3\lambda^3) \\ + \frac{4}{9}(3\lambda^2 - 2\lambda^3 - 2\lambda^4) + \frac{4}{5}\lambda^3 + \frac{2}{11}\lambda^4$$

$$A_3 = \left(\frac{2}{3} - \frac{8}{5} + \frac{12}{7} - \frac{8}{9} + \frac{2}{11}\right)\lambda^4 + \left(\frac{8}{3} - 2 - \frac{24}{5} + 4 + \frac{24}{7} - 3 - \frac{4}{9} + \frac{4}{5}\right)\lambda^3 \\ + \left(4 - 6 - \frac{12}{5} + 8 - \frac{12}{7} - 3 + \frac{4}{9}\right)\lambda^2 + \left(\frac{4}{3} - 6 + \frac{16}{5} + \frac{8}{9} - \frac{24}{7} + 1\right)\lambda \\ + \left(\frac{2}{3} - 2 + \frac{12}{5} - \frac{4}{9} + \frac{2}{7}\right)$$

$$A_3 = 0.07389\lambda^4 + 0.20635\lambda^3 + 0.21904\lambda^2 + 0.10476\lambda + 0.019047$$

$$A_2 = \frac{3}{2}(\lambda^3 + 3\lambda^2 + 3\lambda + 1) - \frac{18}{5}(\lambda^2 + 2\lambda + 1) + 3(-\lambda^3 - 2\lambda^2 + 1) + \frac{6}{7}(6\lambda^2 + 6\lambda - 1) \\ + \frac{9}{4}(\lambda^3 + \lambda^2 - \lambda) - 2\lambda^2 - \frac{3}{5}\lambda^3$$

$$A_2 = \left(\frac{3}{2} - 3 + \frac{9}{4} - \frac{3}{5}\right)\lambda^3 + \left(\frac{9}{2} - \frac{18}{5} - 6 + \frac{36}{7} + \frac{9}{4} - 2\right)\lambda^2 \\ + \left(\frac{9}{2} - \frac{36}{5} + \frac{36}{7} - \frac{9}{4}\right)\lambda + \left(\frac{3}{2} - \frac{18}{5} + 3 - \frac{6}{7}\right)$$

$$A_2 = 0.15000\lambda^3 + 0.292857\lambda^2 + 0.192857\lambda + 0.042857$$

$$A_1 = -\frac{2}{3}(\lambda^2 + 2\lambda + 1) + (1 + \lambda) + \frac{2}{5}[4\lambda^2 + 7\lambda + 9] - \frac{2}{3}(3 + 4\lambda) - \frac{2}{7}(7\lambda^2 + 6\lambda - 3) \\ + \frac{2}{3}\lambda + \frac{2}{3}\lambda^2$$

$$= \left(-\frac{2}{3} + \frac{6}{5} - 2 + \frac{2}{3}\right)\lambda^2 + \left(-\frac{4}{3} + 1 + \frac{14}{5} - \frac{8}{3} - \frac{12}{7} + \frac{2}{2}\right)\lambda + \left(-\frac{2}{3} + 1 + \frac{6}{5} - 2 + \frac{1}{2}\right)$$

$$A_1 = -0.10000\lambda^2 - 0.41429\lambda - 0.009524$$

①	②	③	④	⑤	⑥	⑦	⑧
λ	λ^2	λ^3	λ^4	$0.07369\lambda^5$	$0.20635\lambda^3$	$0.24904\lambda^4$	0.10476λ
-0.5	0.25	-0.125	+0.0625	+0.004618	-0.025794	0.054760	-0.052380
-0.4	0.16	-0.064	+0.0256	+0.001892	-0.013206	0.035046	-0.041904
-0.3	0.09	-0.027	+0.0081	+0.000599	-0.005571	0.019714	-0.031428
-0.2	0.04	-0.008	+0.0016	+0.000118	-0.001651	0.008762	-0.020952
-0.1	0.01	-0.001	+0.0001	+0.000007	-0.000206	0.002190	-0.010476
0	0	0	0	0	0	0	0
+0.1	0.01	+0.001	+0.0001	+0.000007	+0.000206	0.002190	+0.010476
+0.2	0.04	+0.008	+0.0016	+0.000118	+0.001651	0.008762	+0.020952
+0.3	0.09	+0.027	+0.0081	+0.000599	+0.005571	0.019714	+0.031428
+0.4	0.16	+0.064	+0.0256	+0.001892	+0.013206	0.035046	+0.041904
+0.5	0.25	+0.125	+0.0625	+0.004618	+0.025794	0.054760	+0.052380

	⑨	⑩	⑪	⑫	⑬	⑭	⑮
λ	A_3	$0.15311\lambda^3$	$0.27111\lambda^4$	$0.10252\lambda^5$	A_2	$-0.40000\lambda^2$	-0.41429λ
-0.5	+0.000254	-0.01125	0.00014	-0.000007	+0.017767	-0.100000	+0.207145
-0.4	+0.000878	-0.00960	0.046857	-0.000003	+0.011611	-0.064000	+0.165716
-0.3	+0.002364	-0.00405	0.026357	-0.000005	+0.010952	-0.036000	+0.124287
-0.2	+0.005327	-0.00120	0.00214	-0.000002	+0.015880	-0.016000	+0.082858
-0.1	+0.010565	-0.00015	0.002929	-0.000001	+0.026485	-0.004000	+0.041429
0	+0.019050	0	0	0	+0.042857	0	0
+0.1	+0.031929	+0.00015	0.002929	+0.000001	+0.065087	-0.004000	-0.041429
+0.2	+0.050533	+0.00120	0.00214	+0.000002	+0.093262	-0.016000	-0.082858
+0.3	+0.076362	+0.00405	0.026357	+0.000005	+0.127426	-0.036000	-0.124287
+0.4	+0.111098	+0.00960	0.046857	+0.000003	+0.167817	-0.064000	-0.165716
+0.5	+0.156602	+0.01125	0.073214	+0.000007	+0.214375	-0.100000	-0.207145

λ	(16) A_1	(17) A_0	(18) B_1	(19) -2λ	(20) B_0	(21) $k_1 = 4A_1B_1$	(22) B_1A_1
-0.5	+0.097621	-0.010714	0.133333	+1.00	-0.333333	+0.000355	+0.0023689
-0.4	+0.092192	-0.019048	0.245333	+0.8	-0.533333	+0.0008616	+0.0028486
-0.3	+0.078763	-0.027381	0.421333	+0.6	-0.733333	+0.003984	+0.0045184
-0.2	+0.057337	-0.035714	0.661333	+0.4	-0.933333	+0.014092	+0.0105020
-0.1	+0.027905	-0.044048	0.965333	+0.2	-1.133333	+0.040795	+0.0255668
0	-0.009524	-0.052381	1.333333	0	-1.333333	+0.101600	+0.057143
+0.1	-0.054953	-0.060714	1.765333	-0.2	-1.533333	+0.225461	+0.114900
+0.2	-0.108382	-0.069048	2.261333	-0.4	-1.733333	+0.457028	+0.210696
+0.3	-0.169811	-0.077381	2.821333	-0.6	-1.933333	+0.86.722	+0.359652
+0.4	-0.239240	-0.085714	3.445333	-0.8	-2.133333	+1.53108	+0.548185
+0.5	-0.316669	-0.094048	4.125333	-1.0	-2.333333	+2.54915	+0.886083
λ	(23) B_0A_3	(24) K_2	(25) A_1B_0	(26) $E.A_1$	(27) K_3	(28) A_1B_0	(29) B_1A_0
-0.5	-0.0000847	+0.0068526	-0.005922	+0.013016	+0.014188	-0.032540	-0.001429
-0.4	-0.0004683	+0.0021409	-0.006192	+0.022618	+0.030852	-0.049169	-0.004673
-0.3	-0.0017336	+0.0086424	-0.008031	+0.033185	+0.050308	-0.057759	-0.011537
-0.2	-0.0049718	+0.0165906	-0.014821	+0.037917	+0.046192	-0.053512	-0.023619
-0.1	-0.0119736	+0.0407791	-0.030016	+0.026938	-0.006156	-0.031626	-0.042521
0	-0.0253999	+0.095229	-0.0571425	-0.012699	-0.139684	+0.012699	-0.069841
+0.1	-0.0489577	+0.197826	-0.0997978	-0.097010	-0.393820	+0.0842611	-0.107180
+0.2	-0.0875904	+0.369918	-0.161654	-0.245088	-0.813484	+0.187862	-0.156141
+0.3	-0.147633	+0.636057	-0.246453	-0.479093	-1.451070	+0.328301	-0.218318
+0.4	-0.237009	+1.023528	-0.358007	-0.824261	-2.364580	+0.510328	-0.295313
+0.5	-0.365404	+1.562037	-0.500208	-1.30890	-3.618216	+0.738893	-0.386732

$$C_1^3 + \alpha_1 C_1^2 + \alpha_2 C_1 + \alpha_3 = 0$$

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	(30)	(31)	(32)	(33)	(34)	(35)	(36)
λ	K_4	α_1	α_2	α_3	$\xi = -\frac{\alpha_1}{3}$	$-\alpha_1/3$	ρ
-0.5	-0.033969	+50.573	+104.71	-250.69	-16.858	-852.56	-747.85
-0.4	-0.053842	+8.2879	+38.129	-62.491	-27.626	-22.896	+15.233
-0.3	-0.069296	+2.1692	+12.627	-17.393	-0.72307	-1.5685	+11.058
-0.2	-0.077131	+1.1773	+3.2779	-5.4734	-0.39243	-0.46201	+2.8159
-0.1	-0.074447	+0.99962	-0.1509	-1.8176	-0.33321	-0.33308	-0.48398
0	-0.057142	+0.93729	-1.3748	-0.56242	—	—	—
+0.1	-0.02419	+0.87743	-1.7467	-0.10165	-0.29248	-0.25663	-2.0033
+0.2	+0.031721	+0.80929	-1.7797	+0.069398	-0.26976	-0.21631	-1.9980
+0.3	+0.109913	+0.73808	-1.6838	+0.12762	-0.24603	-0.18159	-1.8658
+0.4	+0.245065	+0.66850	-1.5444	+0.14047	-0.22283	-0.14896	-1.6934
+0.5	+0.380161	+0.60330	-1.3975	+0.13524	-0.20110	-0.12132	-1.5188
	(37)	(38)	(39)	(40)	(41)	(42)	(43)
λ	$\frac{2}{9}\alpha_1^2$	(32)-(32)	$-\xi$ (38)	ρ	$-\frac{\rho}{3}$	τ	τ^3
-0.5	568.37	+463.66	7616.4	7565.7			
-0.4	15.267	-22.165	-63.167	-125.66			
-0.3	1.0457	-11.541	-8.3739	-25.767			
-0.2	0.30801	-2.9699	-1.1655	-6.6389			
-0.1	0.22205	+0.37195	+0.12427	-1.6933	+0.16133		
0	—	—	—	—	—		
+0.1	0.17109	+1.9178	+0.56092	+0.45927	+0.66777	+0.81717	+0.54568
+0.2	0.14554	+1.9252	+0.51934	+0.58874	+0.66600	+0.81609	+0.54352
+0.3	0.12106	+1.8049	+0.44406	+0.57168	+0.62180	+0.78854	+0.49031
+0.4	0.09931	+1.6437	+0.36627	+0.50674	+0.56447	+0.75132	+0.42410
+0.5	0.08088	+1.4784	+0.29731	+0.43255	+0.50627	+0.71153	+0.36023

λ	(44) $q/2$	(45) $\cos \theta$	(46) θ'	(47) β_1	(48) β_2	(49) β_3
+0.1	+0.22964	-0.42083	114° 53.2'	38° 17.7'	158° 17.7'	-81° 42.3'
+0.2	+0.29437	-0.54160	122° 47.6'	40° 55.9'	160° 55.9'	-79° 41'
+0.3	+0.28584	-0.58298	125° 39.6'	41° 53.2'	161° 53.2'	-78° 1.8'
+0.4	+0.25337	-0.59743	126° 41.2'	42° 13.7'	162° 13.7'	-77° 46.3'
+0.5	+0.21628	-0.60039	126° 53.9'	42° 18'	162° 18'	-77° 42'
λ	(50) $\cos \beta_1$	(51) $\cos \beta_2$	(52) $\cos \beta_3$	(53) x_1	(54) x_2	(55) x_3
+0.1	0.78483	-0.92910	0.14945	1.28268	-1.51847	0.24425
+0.2	0.75549	-0.94513	0.16964	1.23310	-1.54262	0.30953
+0.3	0.72447	-0.95045	0.20598	1.17409	-1.49894	0.32485
+0.4	0.68147	-0.95228	0.21181	1.11266	-1.43093	0.31677
+0.5	0.63963	-0.95266	0.21203	1.0	1.0	1.0
λ	(56) C_1	(57) C_2	(58) C_1''	(59) B, C_1'''	(60) B, C_1'''	(61) B, C_1'''
+0.1	0.98820	-1.81095	-0.04423	1.74803	-3.19692	-0.08514
+0.2	0.96334	-1.81238	+0.03977	2.17843	-4.09839	+0.08993
+0.3	0.92806	-1.74500	+0.07182	2.61836	-4.92322	+0.22238
+0.4	0.88983	-1.65376	+0.09544	3.06576	-5.69775	+0.32882
+0.5	0.85144	-1.55679	+0.10205	3.51928	-6.43473	+0.42181
λ	(62) + B_0	(63) + B_0	(64) + B_0	$2+3\lambda$	$6/(2+3\lambda)$	
+0.1	+0.24470	-4.73025	-1.61847	2.3	2.6087	
+0.2	+0.44510	-5.83172	-1.64340	2.6	2.3077	
+0.3	+0.68503	-6.85655	-1.71095	2.9	2.0680	
+0.4	+0.93243	-7.83108	-1.80451	3.2	1.8750	
+0.5	+1.18595	-8.76806	-1.91152	3.5	1.7143	

Y = No Use

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λ	(62) $C_1^{(11)^2}$	(63) $C_1^{(12)^2}$	(64) $C_1^{(13)^2}$	(65) $A_3 C_1^{(11)^2}$	(66) $\frac{2}{3} A_2 C_1^{(11)}$	(67) $\frac{1}{5} A_1 + \frac{1}{3} A_2$
0.1	0.98050	3.2795	0.002326	0.031306	0.042966	0.055954
0.2	0.92802	3.2847	0.001589	0.046896	0.059896	0.070665
0.3	0.86130	3.0450	0.006213	0.065771	0.071870	0.086637
0.4	0.79180	2.7349	0.009109	0.087967	0.099552	0.107122
0.5	0.72495	2.4236	0.010414	0.113529	0.121685	0.129658
λ	(68) []	(69) $A_3 C_1^{(11)^2}$	(70) $\frac{2}{3} A_2 C_1^{(11)}$	(71) $\frac{1}{5} A_1 + \frac{1}{3} A_2$	(72) []	(73) $A_2 C_1^{(11)^2}$
0.1	0.037292	0.10471	-0.07858	0.007817	0.005206	0.00207427
0.2		0.16599	-0.11268	0.017183		0.00327994
0.3		0.23252	-0.14830	0.027560		0.0047644
0.4		0.30384	-0.18502	0.039073		0.00610179
0.5		0.37954	-0.22249	0.051474		0.0072739
λ	(74) $\frac{2}{3} A_2 C_1^{(11)} - \frac{1}{5} A_1 + \frac{1}{3} A_2$	(75) []	(76) []	(77) $(76)^{\frac{1}{2}}$	(78) ϕ	(79) $\frac{1}{5} A_1$
0.1	-0.002093	0.020337	0.011520	0.10733	0.4532	
0.2	+0.002473	0.033574	0.014847	0.12185	0.4621	
0.3	+0.006698	0.049432	0.017521	0.13237	0.4686	
0.4	+0.010678	0.068057	0.019753	0.14054	0.4755	
0.5	+0.014585	0.089193	0.021579	0.14690	0.4814	

$$\psi \quad \theta = C \left[\alpha + \left(\frac{1}{C} - 1 \right) \frac{\alpha^5}{\beta^4} \right]$$

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$$\theta^2 - \alpha^2 = \alpha^2 \left[(C^2 - 1) + 2C^2 \left(\frac{1}{C} - 1 \right) \left(\frac{\alpha}{\beta} \right)^4 + C^2 \left(\frac{1}{C} - 1 \right)^2 \left(\frac{\alpha}{\beta} \right)^8 \right]$$

$$\frac{d\theta}{d\alpha} - 1 = (C-1) \left\{ 1 - 5 \left(\frac{\alpha}{\beta} \right)^4 \right\}$$

$$\frac{\theta}{\alpha} - 1 = (C-1) \left\{ 1 - \left(\frac{\alpha}{\beta} \right)^4 \right\}$$

The total potential energy

$$\frac{E(\frac{1}{2})}{4} \int_0^\beta \left\{ (C^2 - 1)^2 + 4C^2 \left(\frac{1}{C} - 1 \right) (C^2 - 1) \left(\frac{\alpha}{\beta} \right)^4 + 2C^2 (3C^2 - 1) \left(\frac{1}{C} - 1 \right)^2 \left(\frac{\alpha}{\beta} \right)^8 + 4C^4 \left(\frac{1}{C} - 1 \right)^3 \left(\frac{\alpha}{\beta} \right)^{12} + C^4 \left(\frac{1}{C} - 1 \right)^4 \left(\frac{\alpha}{\beta} \right)^{16} \right\} \alpha^5 d\alpha$$

$$+ \frac{E(\frac{1}{2})^3}{12} (C-1)^2 \int_0^\beta \left\{ 2 - 12 \left(\frac{\alpha}{\beta} \right)^4 + 24 \left(\frac{\alpha}{\beta} \right)^8 \right\} \alpha d\alpha + \beta C \int_0^\beta \left\{ \alpha + \left(\frac{1}{C} - 1 \right) \frac{\alpha^5}{\beta^4} \right\} d\alpha$$

$$= \frac{E(\frac{1}{2})}{4} \left\{ \frac{(C^2 - 1)^2}{6} + \frac{2}{5} C^2 \left(\frac{1}{C} - 1 \right) (C^2 - 1) + \frac{1}{7} C^2 (3C^2 - 1) \left(\frac{1}{C} - 1 \right)^2 + \frac{2}{9} C^4 \left(\frac{1}{C} - 1 \right)^3 + \frac{1}{22} C^4 \left(\frac{1}{C} - 1 \right)^4 \right\} \beta^6$$

$$+ \frac{E(\frac{1}{2})^3}{12} (C-1)^2 \frac{16}{10} \beta^2 + \beta C \left\{ \frac{1}{4} + \frac{1}{8} \left(\frac{1}{C} - 1 \right) \right\} \beta^4$$

Minimizing with respect to C

$$\frac{E(\frac{1}{2})}{4} \left\{ \frac{2}{3} C (C^2 - 1) - \frac{2}{5} (C-1) (4C^2 + C - 1) + \frac{2}{7} (C-1) (6C^2 - 3C - 1) - \frac{2}{9} (C-1) (4C^2 - 5C + 1) + \frac{2}{11} (C-1)^2 \right\} \beta^6$$

$$+ \frac{E(\frac{1}{2})^3}{12} (C-1) \frac{32}{10} \beta^2 + \beta \frac{1}{8} \beta^4 = 0.$$

$$\rho = 2 E\left(\frac{t}{R}\right) (1-c) \left\{ \rho^2 \left[\frac{2}{3} c(c+1) - \frac{2}{5} (4c^2+c-1) + \frac{2}{7} (6c^2-3c-1) \right. \right. \\ \left. \left. - \frac{2}{9} (4c^2-5c+1) + \frac{2}{11} (c-1)^2 \right] + \frac{32}{30} \left(\frac{t}{R}\right)^2 \rho^{-2} \right\} \quad \underline{24}$$

$$\rho^2 = \frac{4}{\sqrt{15}} \left(\frac{t}{R}\right) [\quad]^{-\frac{1}{2}}$$

$$\frac{\rho}{2 E\left(\frac{t}{R}\right)^2} = \phi = \frac{\rho}{\sqrt{15}} (1-c) \left[\frac{2}{3} (c^2+c) - \frac{2}{5} (4c^2+c-1) + \frac{2}{7} (6c^2-3c-1) \right. \\ \left. - \frac{2}{9} (4c^2-5c+1) + \frac{2}{11} (c-1)^2 \right]^{\frac{1}{2}}$$

$$= \frac{t\sqrt{2}}{\sqrt{15}} (1-c) \frac{1}{3\sqrt{385}} \left[1155(c^2+c) - 693(4c^2+c-1) + 495(6c^2-3c-1) \right. \\ \left. - 385(4c^2-5c+1) + 315(c^2-2c+1) \right]^{\frac{1}{2}}$$

$$= \frac{t}{15} \sqrt{2} \frac{1}{\sqrt{231}} (1-c) \left[128c^2 + 272c + 128 \right]^{\frac{1}{2}}$$

$$= \frac{16\sqrt{2}}{15\sqrt{231}} (1-c) \left[32c^2 + 68c + 32 \right]^{\frac{1}{2}}$$

$$= \frac{32\sqrt{2}}{15\sqrt{231}} (1-c) \left[8c^2 + 17c + 8 \right]^{\frac{1}{2}}$$

$$(1-c)^2 [8c^2 + 17c + 1] = (c-1)^2 [8c^2 + 17c + 1] \quad \underline{\underline{244}}$$

$$+ 2(8c^2 + 17c + 1) + (c-1)(16c + 17)$$

$$\begin{array}{r} 16c^2 + 34c + 16 \\ 16c^2 + 17c \\ - 16c - 17 \\ \hline 32c^2 + 35c - 1 \end{array}$$

$$c = \frac{1}{64} \left[-35 \pm \sqrt{\frac{1225 + 128}{1353}} \right]$$

$$= \frac{1}{64} \left[-35 \pm 36.783 \right] = \begin{array}{l} + \frac{1.783}{64} \\ - \frac{71.783}{64} \end{array} = \begin{array}{l} 0.025643 \\ -1.1216 \end{array}$$

$$8c^2 + 17c + 1 = (1-c) \frac{(16c + 17)}{2} = (1-c) \frac{8.7229}{2} \quad \text{Impe}$$

$$= 0.974357 \times 8.7229$$

$$\phi = 0.974357^{\frac{3}{2}} \times \left(\frac{2241 \times 8.7229}{51975} \right)^{\frac{1}{2}} = \frac{0.59075}{0.21216}$$

$$\text{Let } \theta = C_1 \left[\alpha + \left(\frac{1}{\beta} - 1 \right) \left(\frac{\alpha}{\beta} \right)^2 \right] + C_2 \alpha \left[1 - \left(\frac{\alpha}{\beta} \right)^2 \right] \quad (1)$$

$$= \alpha \left\{ (C_1 + C_2) + (1 - C_1) \left(\frac{\alpha}{\beta} \right)^2 - C_2 \left(\frac{\alpha}{\beta} \right)^4 \right\}$$

$$\theta^2 = \alpha^2 \left\{ (C_1 + C_2)^2 + 2(C_1 + C_2)(1 - C_1) \left(\frac{\alpha}{\beta} \right)^2 + \left[(1 - C_1)^2 - 2C_2(C_1 + C_2) \right] \left(\frac{\alpha}{\beta} \right)^4 - 2C_2(1 - C_1) \left(\frac{\alpha}{\beta} \right)^6 + C_2^2 \left(\frac{\alpha}{\beta} \right)^8 \right\}$$

$$\theta^2 - \alpha^2 = \alpha^2 \left\{ [(C_1 + C_2)^2 - 1] + 2(C_1 + C_2)(1 - C_1) \left(\frac{\alpha}{\beta} \right)^2 + \left[(1 - C_1)^2 - 2C_2(C_1 + C_2) \right] \left(\frac{\alpha}{\beta} \right)^4 - 2C_2(1 - C_1) \left(\frac{\alpha}{\beta} \right)^6 + C_2^2 \left(\frac{\alpha}{\beta} \right)^8 \right\}$$

$$(\theta^2 - \alpha^2)^2 = \alpha^4 \left\{ [(C_1 + C_2)^2 - 1]^2 + 4(C_1 + C_2)(1 - C_1) [(C_1 + C_2)^2 - 1] \left(\frac{\alpha}{\beta} \right)^2 \right.$$

$$+ \left[+ (C_1 + C_2)^2 (1 - C_1)^2 + 2 \left\{ (C_1 + C_2)^2 - 1 \right\} \left[(1 - C_1)^2 - 2C_2(C_1 + C_2) \right] \right] \left(\frac{\alpha}{\beta} \right)^4$$

$$+ \left[4(C_1 + C_2)(1 - C_1) \left\{ (1 - C_1)^2 - 2C_2(C_1 + C_2) \right\} - 4C_2(1 - C_1) [(C_1 + C_2)^2 - 1] \right] \left(\frac{\alpha}{\beta} \right)^6$$

$$+ \left[\left\{ (1 - C_1)^2 - 2C_2(C_1 + C_2) \right\}^2 - 8C_2(1 - C_1)(C_1 + C_2) + 2C_2^2 [(C_1 + C_2)^2 - 1] \right] \left(\frac{\alpha}{\beta} \right)^8$$

$$+ \left[4C_2^2(C_1 + C_2)(1 - C_1) - 4C_2(1 - C_1) \left\{ (1 - C_1)^2 - 2C_2(C_1 + C_2) \right\} \right] \left(\frac{\alpha}{\beta} \right)^{10}$$

$$+ \left[4C_2^2(1 - C_1)^2 + 2C_2^2 \left\{ (1 - C_1)^2 - 2C_2(C_1 + C_2) \right\} \right] \left(\frac{\alpha}{\beta} \right)^{12}$$

$$+ \left[-4C_2^3(1-C_1) \left(\frac{\alpha}{\rho} \right)^{14} + C_2^4 \left(\frac{\alpha}{\rho} \right)^{16} \right]$$

(2)

$$\int_0^1 (1-\alpha^2)^2 d\alpha = \left\{ \frac{1}{6} [(C_1+C_2)^2-1]^2 + \frac{1}{2} (C_1+C_2)(1-C_1) [(C_1+C_2)^2-1] \right. \\ + \frac{2}{5} (C_1+C_2)^2 (1-C_1)^2 + \frac{1}{5} [(C_1+C_2)^2-1] [(1-C_1)^2 - 2C_2(C_1+C_2)] \\ + \frac{1}{3} (C_1+C_2)(1-C_1) [(1-C_1)^2 - 2C_2(C_1+C_2)] - \frac{1}{3} C_2(1-C_1) [(C_1+C_2)^2-1] \\ + \frac{1}{14} [(1-C_1)^2 - 2C_2(C_1+C_2)]^2 - \frac{4}{7} C_2(1-C_1)^2 (C_1+C_2) + \frac{1}{7} C_2^2 [(C_1+C_2)^2-1] \\ + \frac{1}{4} C_2^2 (C_1+C_2)(1-C_1) - \frac{1}{4} C_2(1-C_1) [(1-C_1)^2 - 2C_2(C_1+C_2)] \\ + \frac{2}{9} C_2^2 (1-C_1)^2 + \frac{1}{9} C_2^2 [(1-C_1)^2 - 2C_2(C_1+C_2)] - \frac{1}{5} C_2^3 (1-C_1) \\ \left. + \frac{1}{22} C_2^4 \right\}$$

put $(C_1+C_2)=a, (1-C_1)=b, C_2=c$

$$= \frac{1}{6} (a^2-1)^2 + \frac{1}{2} ab(a^2-1) + \frac{2}{5} a b^2 + \frac{1}{5} (a^2-1)(b^2-2ac) \\ + \frac{1}{3} ab(b^2-2ac) - \frac{1}{3} cb(a^2-1) + \frac{1}{14} (b^2-2ac)^2 - \frac{4}{7} c b^2 a \\ + \frac{1}{7} c^2(a^2-1) + \frac{1}{4} c^2 ab - \frac{1}{4} cb(b^2-2ac) + \frac{2}{9} c^2 b^2 + \frac{1}{9} c^2(b^2-2ac) \\ - \frac{1}{5} c^3 b + \frac{1}{22} c^4$$

$$\begin{aligned}
&= \frac{1}{6}(a^4 - 2a^2 + 1) + \frac{1}{2}(a^3b - ab) + \frac{1}{5}a^2b^2 + \frac{1}{5}(a^2b^2 - 2a^2c - b^2 + 2ac) \\
&+ \frac{1}{3}(ab^3 - 2a^2bc) - \frac{1}{3}(a^2bc - bc) + \frac{1}{14}(b^4 - 4ab^2c + 4a^2c^2) \\
&- \frac{4}{7}ab^2c - \frac{1}{7}(a^2c^2 - c^2) + \frac{1}{4}abc^2 - \frac{1}{4}(b^3c - 2ab^2c^2) + \frac{2}{7}b^2c^2 \\
&+ \frac{1}{9}(b^2c^2 - 2ac^3) - \frac{1}{5}bc^3 + \frac{1}{12}c^4
\end{aligned}$$

$$\begin{aligned}
&= a^4\left(\frac{1}{6}\right) + a^3b\left(\frac{1}{2}\right) + a^2b^2\left(\frac{2}{5} + \frac{1}{5}\right) + ab^3\left(\frac{1}{3}\right) + b^4\left(\frac{1}{14}\right) \\
&+ a^3c\left(-\frac{2}{5}\right) + a^2c^2\left(\frac{2}{7} - \frac{1}{7}\right) + ac^3\left(-\frac{2}{9}\right) + c^4\left(\frac{1}{12}\right) \\
&+ b^3c\left(-\frac{1}{4}\right) + b^2c^2\left(\frac{2}{9} + \frac{1}{9}\right) + bc^3\left(-\frac{1}{5}\right) + a^2bc\left(-\frac{2}{3} - \frac{1}{3}\right) \\
&+ ab^2c\left(-\frac{2}{7} - \frac{4}{7}\right) + abc^2\left(\frac{1}{4} + \frac{1}{2}\right) - \frac{1}{3}a^2 + \frac{1}{6} - \frac{1}{2}ab \\
&- \frac{1}{5}b^2 + \frac{2}{5}ac + \frac{1}{3}bc + \frac{1}{7}c^2
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{6}a^4 + \frac{1}{2}a^3b + \frac{3}{5}a^2b^2 + \frac{1}{3}ab^3 + \frac{1}{14}b^4 - \frac{2}{5}a^3c + \frac{1}{7}a^2c^2 \\
&- \frac{2}{9}ac^3 + \frac{1}{12}c^4 - \frac{1}{4}b^3c + \frac{1}{3}b^2c^2 - \frac{1}{5}bc^3 - a^2bc - \frac{6}{7}ab^2c \\
&+ \frac{3}{4}abc^2 - \frac{1}{3}a^2 - \frac{1}{2}ab - \frac{1}{5}b^2 + \frac{2}{5}ac + \frac{1}{3}bc + \frac{1}{7}c^2 + \frac{1}{6}
\end{aligned}$$

(4)

$$\begin{aligned}
& - \frac{1}{6} (C_1^4 + 4C_1^3 C_2 + 6C_1^2 C_2^2 + 4C_1 C_2^3 + C_2^4) \\
& + \frac{1}{2} (C_1^3 + 3C_1^2 C_2 + 3C_1 C_2^2 + C_2^3 - C_1^4 - 3C_1^3 C_2 - 3C_1^2 C_2^2 - C_1 C_2^3) \\
& + \frac{3}{5} (C_1^4 + 2C_1^3 C_2 + C_1^2 C_2^2 - 2C_1^3 - 4C_1^2 C_2 - 2C_1 C_2^2 + C_1^2 + 2C_1 C_2 + C_2^2) \\
& + \frac{1}{3} (C_1^3 - 3C_1^2 + 3C_1^3 - C_1^4 + C_2 - 3C_1 C_2 + 3C_1^2 C_2 - C_1^3 C_2) \\
& + \frac{1}{14} (1 - 4C_1 + 6C_1^2 - 4C_1^3 + C_1^4) \\
& - \frac{2}{5} (C_1^3 C_2 + 3C_1^2 C_2^2 + 3C_1 C_2^3 + C_2^4) \\
& + \frac{1}{7} (C_1^2 C_2^2 + 2C_1 C_2^3 + C_2^4) \\
& - \frac{2}{9} (C_1 C_2^3 + C_2^4) + \frac{1}{22} C_2^4 \\
& - \frac{1}{4} (C_2 - 3C_1 C_2 + 3C_1^2 C_2 - C_1^3 C_2) \\
& + \frac{1}{3} (C_2^2 - 2C_1 C_2^2 + C_1^2 C_2^2) - \frac{1}{5} (C_2^3 - C_1 C_2^3) \\
& - (C_1^2 C_2 + 2C_1 C_2^2 + C_2^3 - C_1^3 C_2 - 2C_1^2 C_2^2 - C_1 C_2^3) \\
& - \frac{6}{7} (C_1 C_2 - 2C_1^2 C_2 + C_1^3 C_2 + C_2^2 - 2C_1 C_2^2 + C_1^2 C_2^2) \\
& + \frac{3}{4} (-C_1^2 C_2^2 - C_1 C_2^3 + C_1 C_2^2 + C_2^3) - \frac{1}{3} (C_1^2 + 2C_1 C_2 + C_2^2) \\
& - \frac{1}{2} (-C_1^2 - C_1 C_2 + C_1 + C_2) - \frac{1}{5} (1 - 2C_1 + C_1^2)
\end{aligned}$$

$$\begin{aligned}
& + \frac{2}{5} (C_1^2 C_2 + C_2^2) + \frac{1}{3} (C_2 - C_1 C_2) + \frac{1}{7} C_2^2 + \frac{1}{6} \quad (5) \\
& = C_1^4 \left(\frac{1}{6} - \frac{1}{2} + \frac{3}{5} - \frac{1}{3} + \frac{1}{14} \right) + C_1^3 C_2 \left(\frac{2}{3} - \frac{3}{2} + \frac{6}{5} - \frac{1}{3} - \frac{2}{5} + \frac{1}{4} + \frac{1}{-7} \right) \\
& + C_1^2 C_2^2 \left(1 - \frac{2}{2} + \frac{3}{5} - \frac{6}{5} + \frac{1}{7} + \frac{1}{3} + 1 - \frac{6}{7} - \frac{2}{4} \right) \\
& + C_1 C_2^3 \left(\frac{2}{3} - \frac{1}{2} - \frac{6}{5} + \frac{2}{7} - \frac{2}{9} + \frac{1}{5} + 1 - \frac{3}{4} \right) \\
& + C_2^4 \left(\frac{1}{6} - \frac{2}{5} + \frac{1}{7} + \frac{1}{22} - \frac{2}{9} \right) \\
& + C_1^3 \left(\frac{1}{2} - \frac{1}{5} + 1 - \frac{2}{7} \right) \\
& + C_1^2 C_2 \left(\frac{3}{2} - \frac{12}{5} + 1 - \frac{3}{4} - 1 + \frac{12}{7} \right) \\
& + C_1 C_2^2 \left(\frac{3}{2} - \frac{6}{5} - \frac{2}{3} - 2 + \frac{12}{7} + \frac{2}{4} \right) \\
& + C_2^2 \left(\frac{1}{2} - \frac{1}{5} - 1 + \frac{2}{7} \right) \\
& + C_1^2 \left(\frac{3}{5} - 1 + \frac{3}{7} - \frac{1}{3} + \frac{1}{2} - \frac{1}{5} \right) \\
& + C_1 C_2 \left(\frac{6}{5} - 1 + \frac{3}{4} - \frac{1}{7} - \frac{2}{3} + \frac{1}{2} + \frac{2}{5} - \frac{1}{3} \right) \\
& + C_2^2 \left(\frac{3}{5} + \frac{1}{3} - \frac{6}{7} - \frac{1}{3} + \frac{2}{5} + \frac{1}{7} \right) \\
& + C_1 \left(\frac{1}{3} - \frac{2}{7} - \frac{1}{2} + \frac{2}{5} \right) + C_2 \left(-\frac{1}{4} - \frac{1}{2} + \frac{1}{3} + \frac{1}{3} \right) \\
& + \left(\frac{1}{14} - \frac{1}{5} + \frac{1}{6} \right)
\end{aligned}$$

85
terms

$$\int_0^{\beta} (1-\alpha^2)^2 \alpha d\alpha = \beta^6 \left[\frac{1}{210} C_1^6 + \frac{11}{420} C_1^3 C_2 - \frac{97}{420} C_1^2 C_2^2 \right. \\ \left. - \frac{131}{252} C_1 C_2^3 - \frac{926}{3465} C_2^4 + \frac{1}{70} C_1^3 + \frac{9}{140} C_1^2 C_2 + \frac{41}{420} C_1 C_2^2 \right. \\ \left. + \frac{1}{20} C_2^3 - \frac{1}{110} C_1^2 - \frac{1}{140} C_1 C_2 + \frac{2}{7} C_2^2 + \frac{11}{210} C_1 - \frac{1}{12} C_2 + \frac{1}{210} \right] \quad (6)$$

$$\frac{dQ}{d\alpha} = (C_1 + C_2) + 3(1-C_1)\left(\frac{\alpha}{\beta}\right)^2 - 5C_2\left(\frac{\alpha}{\beta}\right)^4$$

$$\left(\frac{dQ}{d\alpha} - 1\right)^2 = (C_1 + C_2 - 1)^2 + 6(1-C_1)(C_1 + C_2 - 1)\left(\frac{\alpha}{\beta}\right)^2 \\ + \left[9(1-C_1)^2 - 10C_2(C_1 + C_2 - 1)\right]\left(\frac{\alpha}{\beta}\right)^4 - 30C_2(1-C_1)\left(\frac{\alpha}{\beta}\right)^6 + 25C_2^2\left(\frac{\alpha}{\beta}\right)^8$$

$$\left(\frac{Q}{\alpha} - 1\right)^2 = (C_1 + C_2 - 1)^2 + 2(1-C_1)(C_1 + C_2 - 1)\left(\frac{\alpha}{\beta}\right)^2 \\ + \left[(1-C_1)^2 - 2C_2(C_1 + C_2 - 1)\right]\left(\frac{\alpha}{\beta}\right)^4 - 2C_2(1-C_1)\left(\frac{\alpha}{\beta}\right)^6 + C_2^2\left(\frac{\alpha}{\beta}\right)^8$$

$$\int_0^{\beta} \left\{ \left(\frac{dQ}{d\alpha} - 1\right)^2 + \left(\frac{Q}{\alpha} - 1\right)^2 \right\} \alpha d\alpha$$

$$= \beta^2 \left[(C_1 + C_2 - 1)^2 + 2(1-C_1)(C_1 + C_2 - 1) + \frac{5}{3}(1-C_1)^2 - 2C_2(C_1 + C_2 - 1) \right. \\ \left. - 4C_2(1-C_1) + 2.6 C_2^2 \right]$$

$$\int_0^\beta \Theta \alpha^2 d\alpha = \beta^3 \left[\frac{1}{4}(C_1 + C_2) + \frac{1}{6}(1 - C_1) - \frac{1}{8}C_2 \right] \\ - \beta^4 \left[\frac{1}{6} + \frac{1}{12}C_1 + \frac{1}{8}C_2 \right]$$

(7)

$$\int_0^\beta \left\{ \left(\frac{d\Theta}{d\alpha} - 1 \right)^2 + \left(\frac{\Theta}{\alpha} - 1 \right)^2 \right\} \alpha d\alpha$$

$$= \beta^3 \left[\frac{2}{3}C_1^2 + 2C_1C_2 + 1.6C_2^2 - \frac{4}{3}C_1 - 2C_2 + \frac{2}{3} \right]$$

$$z_0 = R \int_0^\beta \Theta d\alpha = R\beta^2 \left[\frac{1}{2}(C_1 + C_2) + \frac{1}{4}(1 - C_1) - \frac{1}{6}C_2 \right]$$

$$= R\beta^2 \left[\frac{1+C_1}{4} + \frac{1}{3}C_2 \right]$$

$$S = R\beta^2 \left[\frac{1-C_1}{4} - \frac{1}{3}C_2 \right]$$

$$\frac{S}{z} = w = \left(\frac{R}{L} \right) \beta^2 \left[\frac{1}{4}(1 - C_1) - \frac{1}{3}C_2 \right]$$

The total energy

(d)

$$E\left(\frac{p}{R}\right) \beta^6 \left\{ \frac{1}{210} C_1^4 + \frac{11}{420} C_1^3 C_2 - \frac{97}{420} C_1^2 C_2^2 - \frac{131}{252} C_1 C_2^3 - \frac{921}{3465} C_2^4 + \frac{1}{70} C_1^3 + \frac{9}{140} C_1^2 C_2 \right. \\ \left. + \frac{41}{420} C_1 C_2^2 + \frac{1}{20} C_2^3 - \frac{1}{210} C_1^2 - \frac{1}{140} C_1 C_2 + \frac{2}{7} C_2^2 - \frac{11}{210} C_1 - \frac{1}{12} C_2 + \frac{2}{210} \right\}$$

$$+ E\left(\frac{p}{R}\right)^3 \beta^3 \left\{ \frac{2}{9} C_1^2 + \frac{2}{3} C_1 C_2 + \frac{8}{15} C_2^2 - \frac{4}{9} C_1 - \frac{2}{3} C_2 + \frac{2}{9} \right\}$$

$$+ p \beta^4 \left\{ \frac{2}{3} + \frac{1}{3} C_1 + \frac{1}{2} C_2 \right\}$$

$$\text{Put } \sigma = \frac{p}{2} \frac{1}{\left(\frac{p}{R}\right)}$$

$$K = \frac{\sigma}{E\left(\frac{p}{R}\right)} = \frac{p}{2} \frac{1}{E\left(\frac{p}{R}\right)^2}$$

$$\frac{\sigma^2}{\left(\frac{p}{R}\right)} = \gamma^2$$

$$\beta^6 \left\{ \frac{1}{210} C_1^4 + \frac{11}{420} C_1^3 C_2 - \frac{97}{420} C_1^2 C_2^2 - \frac{131}{252} C_1 C_2^3 - \frac{921}{3465} C_2^4 + \frac{1}{70} C_1^3 + \frac{9}{140} C_1^2 C_2 \right. \\ \left. + \frac{41}{420} C_1 C_2^2 + \frac{1}{20} C_2^3 - \frac{1}{210} C_1^2 - \frac{1}{140} C_1 C_2 + \frac{2}{7} C_2^2 - \frac{11}{210} C_1 - \frac{1}{12} C_2 - \frac{2}{210} \right\}$$

$$+ \gamma^2 \left\{ \frac{2}{9} C_1^2 + \frac{2}{3} C_1 C_2 + \frac{8}{15} C_2^2 - \frac{4}{9} C_1 - \frac{2}{3} C_2 + \frac{2}{9} \right\}$$

$$+ K \gamma^4 \left\{ \frac{4}{3} + \frac{2}{3} C_1 + C_2 \right\}$$

$$W = \gamma^2 \left\{ \frac{1}{4}(1-C_1) - \frac{1}{3} C_2 \right\}$$

$$\gamma^4 \left\{ \frac{2}{105} C_1^3 + \frac{11}{140} C_1^2 C_2 - \frac{97}{210} C_1 C_2^2 - \frac{131}{252} C_2^3 + \frac{3}{70} C_1^2 + \frac{9}{70} C_1 C_2 + \frac{41}{420} C_2^2 \right. \\ \left. - \frac{1}{105} C_1 - \frac{1}{140} C_2 - \frac{11}{210} \right\} + \left\{ \frac{4}{9} C_1 + \frac{2}{3} C_2 - \frac{4}{9} \right\} + K \gamma^2 \left(\frac{2}{3} \right) = 0 \quad (8)$$

$$\gamma^4 \left\{ \frac{11}{420} C_1^3 - \frac{97}{210} C_1^2 C_2 - \frac{131}{84} C_1 C_2^2 - \frac{3704}{3465} C_2^3 + \frac{9}{140} C_1^2 + \frac{41}{210} C_1 C_2 + \frac{3}{20} C_2^2 \right. \\ \left. - \frac{1}{140} C_1 + \frac{4}{7} C_2 - \frac{1}{12} \right\} + \left\{ \frac{2}{3} C_1 + \frac{16}{15} C_2 - \frac{2}{3} \right\} + K \gamma^2 = 0.$$

$$2\gamma^3 \left\{ \frac{2}{105} C_1^3 + \frac{11}{140} C_1^2 C_2 - \frac{97}{210} C_1 C_2^2 - \frac{131}{252} C_2^3 + \frac{3}{70} C_1^2 + \frac{9}{70} C_1 C_2 + \frac{41}{420} C_2^2 - \right. \\ \left. - \frac{1}{105} C_1 - \frac{1}{140} C_2 - \frac{11}{210} \right\} + \frac{2}{3} K \\ + \gamma^4 \left\{ \frac{2}{35} C_1^2 \frac{dC_1}{d\gamma} + \frac{11}{140} (2C_1 C_2 \frac{dC_2}{d\gamma} + C_1^2 \frac{dC_1}{d\gamma}) - \frac{97}{210} (C_2^2 \frac{dC_2}{d\gamma} + 2C_1 C_2 \frac{dC_1}{d\gamma}) \right. \\ \left. - \frac{131}{84} C_2^2 \frac{dC_2}{d\gamma} + \frac{3}{35} C_1 \frac{dC_1}{d\gamma} + \frac{9}{70} (C_2 \frac{dC_1}{d\gamma} + C_1 \frac{dC_2}{d\gamma}) + \frac{41}{210} C_2 \frac{dC_2}{d\gamma} \right. \\ \left. - \frac{1}{105} \frac{dC_1}{d\gamma} - \frac{1}{140} \frac{dC_2}{d\gamma} \right\} + \left\{ \frac{4}{9} \frac{dC_1}{d\gamma} + \frac{2}{3} \frac{dC_2}{d\gamma} \right\} = 0$$

$$- \frac{2}{\gamma^2} \left\{ \frac{4}{9} C_1 + \frac{2}{3} C_2 - \frac{4}{9} \right\} - \frac{2}{3} K \\ + \gamma^4 \left\{ \left(\frac{2}{35} C_1^2 + \frac{11}{70} C_1 C_2 - \frac{97}{210} C_2^2 + \frac{3}{35} C_1 + \frac{9}{70} C_2 - \frac{1}{105} \right) \frac{dC_1}{d\gamma} \right. \\ \left. + \left(\frac{11}{140} C_1^2 - \frac{97}{105} C_1 C_2 - \frac{131}{84} C_2^2 + \frac{9}{70} C_1 + \frac{41}{210} C_2 - \frac{1}{140} \right) \frac{dC_2}{d\gamma} \right\} \\ + \frac{4}{9} \frac{dC_1}{d\gamma} + \frac{2}{3} \frac{dC_2}{d\gamma} = 0$$

(12)

$$\begin{aligned}
& -\frac{2}{j^2} \left\{ \frac{2}{3} C_1 + \frac{16}{15} C_2 - \frac{2}{3} \right\} - K \\
& + j^4 \left\{ \left(\frac{11}{140} C_1^2 - \frac{91}{105} C_1 C_2 - \frac{131}{84} C_2^2 + \frac{9}{70} C_1 + \frac{41}{210} C_2 - \frac{1}{140} \right) \frac{dC_1}{dy} \right. \\
& \quad \left. + \left(-\frac{91}{210} C_1^2 - \frac{131}{42} C_1 C_2 - \frac{3704}{1155} C_2^2 + \frac{41}{210} C_1 + \frac{3}{10} C_2 + \frac{4}{7} \right) \frac{dC_2}{dy} \right\} \\
& + \left\{ \frac{2}{3} \frac{dC_1}{dy} + \frac{16}{15} \frac{dC_2}{dy} \right\} = 0
\end{aligned}$$

$$0 = \frac{1}{4} - \frac{1}{4} C_1 - \frac{1}{3} C_2 - j^2 \left(\frac{1}{4} \frac{dC_1}{dy} + \frac{1}{3} \frac{dC_2}{dy} \right)$$

$$\therefore \frac{1}{3} j^2 \frac{dC_2}{dy} = \left(\frac{1}{4} - \frac{1}{4} C_1 - \frac{1}{3} C_2 \right) - \frac{1}{4} j^2 \frac{dC_1}{dy}$$

$$\begin{aligned}
& -\frac{2}{j^2} \left\{ \frac{4}{9} C_1 + \frac{2}{3} C_2 - \frac{2}{9} \right\} - \frac{2}{3} K \\
& + j^2 \left(\frac{1}{4} - \frac{1}{4} C_1 - \frac{1}{3} C_2 \right) \left(\frac{33}{140} C_1^2 - \frac{71}{70} C_1 C_2 - \frac{131}{84} C_2^2 + \frac{21}{70} C_1 + \frac{41}{210} C_2 - \frac{1}{140} \right) \\
& + j^4 \left(-\frac{1}{510} C_1^2 + \frac{119}{140} C_1 C_2 + \frac{1189}{1680} C_2^2 - \frac{3}{280} C_1 - \frac{1}{56} C_2 - \frac{1}{240} \right) \frac{dC_1}{dy} \\
& + \frac{1}{j^2} \left(\frac{1}{2} - \frac{1}{2} C_1 - \frac{2}{3} C_2 \right) - \frac{1}{18} \frac{dC_1}{dy} = 0
\end{aligned}$$

(11)

$$\begin{aligned}
& -\frac{2}{y^2} \left\{ \frac{2}{3} C_1 + \frac{16}{15} C_2 - \frac{2}{3} \right\} - K \\
& + y^2 \left(\frac{1}{4} - \frac{1}{4} C_1 - \frac{1}{3} C_2 \right) \left(-\frac{97}{70} C_1^2 - \frac{131}{14} C_1 C_2 - \frac{3764}{385} C_2^2 + \frac{41}{70} C_1 + \frac{9}{10} C_2 + \frac{12}{7} \right) \\
& + y^4 \left(\frac{119}{280} C_1^2 + \frac{1189}{840} C_1 C_2 + \frac{3907}{4620} C_2^2 - \frac{1}{56} C_1 - \frac{5}{168} C_2 - \frac{61}{140} \right) \frac{dC}{dy} \\
& + \frac{1}{y^2} \left(\frac{4}{5} - \frac{4}{5} C_1 - \frac{16}{15} C_2 \right) - \frac{2}{15} \frac{dC}{dy} = 0
\end{aligned}$$

$$\begin{aligned}
& \left(\frac{1}{4} - \frac{1}{4} C_1 - \frac{1}{3} C_2 \right) \left(\frac{33}{140} C_1^2 - \frac{27}{35} C_1 C_2 - \frac{131}{28} C_2^2 + \frac{21}{70} C_1 + \frac{41}{70} C_2 - \frac{3}{140} \right) \\
& = \\
& \quad + \frac{33}{560} C_1^2 - \frac{97}{140} C_1 C_2 - \frac{131}{112} C_2^2 + \frac{21}{280} C_1 + \frac{41}{280} C_2 - \frac{3}{560} \\
& - \frac{33}{560} C_1^3 + \frac{97}{140} C_1^2 C_2 + \frac{131}{112} C_1 C_2^2 + 0 - \frac{21}{280} C_1^2 - \frac{41}{280} C_1 C_2 + 0 + \frac{3}{560} C_1 + 0 + 0 \\
& \quad - \frac{11}{140} C_1^2 C_2 + \frac{97}{105} C_1 C_2^2 + \frac{131}{84} C_2^3 + 0 - \frac{9}{70} C_1 C_2 - \frac{41}{210} C_2^2 + 0 + \frac{1}{140} C_2 + 0 \\
& = -\frac{33}{560} C_1^3 + \frac{86}{140} C_1^2 C_2 + \frac{3577}{1680} C_1 C_2^2 + \frac{131}{84} C_2^3 - \frac{3}{80} C_1^2 - \frac{271}{280} C_1 C_2 - \dots
\end{aligned}$$

Too Complicated

$$y^4 \left\{ \frac{1}{35} C_1^3 + \frac{33}{280} C_1^2 C_2 - \frac{97}{140} C_1 C_2^2 - \frac{131}{168} C_2^3 + \frac{9}{140} C_1^2 + \frac{27}{140} C_1 C_2 + \frac{41}{280} C_2^2 \right. \\ \left. - \frac{1}{70} C_1 - \frac{3}{280} C_2 - \frac{11}{140} \right\} + \left\{ \frac{2}{3} C_1 + C_2 - \frac{2}{3} \right\} + K y^2 = 0$$

$$y^4 \left\{ \frac{1}{420} C_1^3 + \frac{441}{840} C_1^2 C_2 + \frac{13}{15} C_1 C_2^2 + \frac{8017}{24720} C_2^3 + 0 - \frac{1}{420} C_1 C_2 - \frac{1}{40} C_2^2 \right. \\ \left. - \frac{1}{140} C_1 - \frac{163}{40} C_2 + \frac{1}{240} \right\} - \frac{1}{15} C_2 = 0$$

$$\frac{d}{dx}\left(x \frac{dv}{dx}\right) = \frac{v}{x} \left\{ 1 + \Lambda x^2 (v+x)(v+2x) \right\} + x^2$$

The boundary conditions: $x=0, v=0$

$$x=a, v=0$$

Starting with the function $v = Cx \left[1 - \left(\frac{x}{a}\right)^2 \right]$. Then

$$\begin{aligned} \frac{d}{dx}\left(x \frac{dv}{dx}\right) &= C \left[1 - \left(\frac{x}{a}\right)^2 \right] \left\{ 1 + \Lambda x^2 \left[(C+1)x - Cx \left(\frac{x}{a}\right)^2 \right] \left[(C+2)x - Cx \left(\frac{x}{a}\right)^2 \right] \right\} \\ &= C \left[1 - \left(\frac{x}{a}\right)^2 \right] + C \Lambda x^2 \left[1 - \left(\frac{x}{a}\right)^2 \right] \left[(C+1)x - Cx \left(\frac{x}{a}\right)^2 \right] \left[(C+2)x - Cx \left(\frac{x}{a}\right)^2 \right] \\ &\quad + x^2 \end{aligned}$$

$$= C \left[1 - \left(\frac{x}{a}\right)^2 \right] + C \Lambda \left\{ (C+1)(C+2)x^4 - (2C^2 + 5C + 4)x^6 \left(\frac{x}{a}\right)^2 + 2C(C+2)x^4 \left(\frac{x}{a}\right)^4 - C^2 x^6 \left(\frac{x}{a}\right)^6 \right\} + x^2$$

$$\begin{aligned} \frac{dv}{dx} &= C \left[1 - \frac{1}{3} \left(\frac{x}{a}\right)^2 \right] + C \Lambda \left\{ \frac{(C+1)(C+2)}{5} x^4 - \frac{(2C^2 + 5C + 4)}{7} x^6 \left(\frac{x}{a}\right)^2 \right. \\ &\quad \left. + \frac{2}{9} C(C+2) x^4 \left(\frac{x}{a}\right)^4 - \frac{C^2}{11} x^6 \left(\frac{x}{a}\right)^6 \right\} + \frac{1}{3} x^2 \end{aligned}$$

$$\begin{aligned} v &= Cx \left[1 - \frac{1}{9} \left(\frac{x}{a}\right)^2 \right] + C \Lambda \left\{ \frac{C+1}{25} - \frac{(2C^2 + 5C + 4)}{49} \left(\frac{x}{a}\right)^2 \right. \\ &\quad \left. + \frac{2}{81} C(C+2) \left(\frac{x}{a}\right)^4 - \frac{C^2}{121} \left(\frac{x}{a}\right)^6 \right\} x^5 + \frac{1}{9} x^3 \end{aligned}$$

$$\begin{aligned} \therefore 1 - \left(\frac{x}{a}\right)^2 &= \left[1 - \frac{1}{9} \left(\frac{x}{a}\right)^2 \right] + \Lambda \left\{ \frac{(C+1)(C+2)}{25} - \frac{(2C^2 + 5C + 4)}{49} \left(\frac{x}{a}\right)^2 \right. \\ &\quad \left. + \frac{2}{81} C(C+2) \left(\frac{x}{a}\right)^4 - \frac{C^2}{121} \left(\frac{x}{a}\right)^6 \right\} x^4 + \frac{1}{9} \frac{x^2}{C} \end{aligned}$$

$$\begin{aligned}
 -\frac{3}{4} &= \left[1 - \frac{1}{36}\right] + \Delta \left\{ \frac{(C+1)(C+2)}{25} - \frac{(2C^2+5C+4)}{196} + \frac{1}{81 \times 8} C(C+2) \right. \\
 &\quad \left. - \frac{C^2}{121} \frac{1}{64} \right\} \frac{a^4}{16} + \frac{1}{36} \frac{a^2}{C} \\
 &= \left\{ \frac{(C+1)(C+2)}{25} - \frac{(2C^2+5C+4)}{196} + \frac{C(C+2)}{81 \times 8} - \frac{C^2}{64 \times 121} \right\} \Delta \\
 &= -\frac{2}{9} \frac{16}{a^4} - \frac{1}{36} \frac{1}{C} \frac{16}{a^2} = -\frac{32}{9} \frac{1}{a^4} - \frac{4}{9} \frac{1}{C} \frac{1}{a^2} \\
 &= -\frac{4}{9} \left[-8 \frac{1}{a^4} - \frac{1}{C} \frac{1}{a^2} \right]
 \end{aligned}$$

$$\frac{\partial \Delta}{\partial a^2} = 0, \quad \frac{16}{a^6} + \frac{1}{C} \frac{1}{a^4} = 0$$

$$\frac{16}{a^2} + \frac{1}{C} = 0 \quad -16C = a^2$$

Then

$$\begin{aligned}
 &\left\{ \frac{(C+1)(C+2)}{25} - \frac{(2C^2+5C+4)}{196} + \frac{C(C+2)}{81 \times 8} - \frac{C^2}{64 \times 121} \right\} \Delta \\
 &= \frac{4}{9} \left[-\frac{8}{16^2} \frac{1}{C^2} + \frac{1}{C^2} \frac{1}{16} \right] \\
 &= \frac{1}{36} \frac{1}{C^2} \frac{1}{2} = \frac{1}{72} \frac{1}{C^2}
 \end{aligned}$$

$$\left[\frac{(C+1)(C+2)}{25} - \frac{(2C^2+5C+4)}{196} + \frac{C(C+2)}{8 \times 81} - \frac{C^2}{64 \times 121} \right] \frac{1}{\phi^2}$$

$$= \frac{1}{3} \frac{1}{C^2}$$

$$\phi = \sqrt{3} C \left[\frac{(C+1)(C+2)}{25} - \frac{(2C^2+5C+4)}{196} + \frac{C(C+2)}{8 \times 81} - \frac{C^2}{64 \times 121} \right]^{\frac{1}{2}}$$

$$= \frac{\sqrt{3} C}{5 \times 7 \times 9 \times 11 \times 8} \left[49 \times 81 \times 121 \times 64 (C^2 + 3C + 2) \right.$$

$$- 25 \times 81 \times 121 \times 16 (2C^2 + 5C + 4) + 25 \times 49 \times 121 \times 8 (C^2 + 2C)$$

$$\left. - 25 \times 49 \times 81 C^2 \right]^{\frac{1}{2}}$$

$$= \frac{\sqrt{3} C}{5 \times 2 \times 9 \times 11 \times 8} \left[2398171 C^2 + 74972408 C + 45790272 \right]^{\frac{1}{2}}$$

$$= \frac{\sqrt{3 \times 2398171} C}{5 \times 2 \times 9 \times 11 \times 8} \left[C^2 + 3.12644 C + 1.90938 \right]^{\frac{1}{2}}$$

$$= \frac{\sqrt{3} C}{5} \left[C^2 + 3.12644 C + 1.90938 \right]^{\frac{1}{2}}$$

$$2[C^2 + 3.12644 C + 1.90938] + C[2C + 3.12644] = 0$$

$$C^2 + 3.12644 C + 1.90938$$

$$C^2 + 1.56322 C + 0$$

$$C^2 + 2.34483 C + 0.95469$$

$$C = -1.17242 \pm \sqrt{1.37457 - 0.95469}$$

$$= -1.17242 \pm \sqrt{0.41988} = -1.17242 \pm 0.648298$$

$$C = \begin{matrix} -0.5244 \\ -1.82040 \end{matrix} \Bigg\}$$

$$C^2 + 3.12144C + 1.9938 = -\frac{C}{2} [2C + 3.12144]$$

$$= -C [C + 1.56322] = +0.4617$$

Two sig.

The differential equation

$$x^2 \frac{d^2 \Theta}{dx^2} + x \frac{d\Theta}{dx} - \Theta = C_1 x^3 + C_2 x^2 \Theta (\Theta^2 - x^2)$$

Put $x = \lambda x$

$\Theta = \lambda y$

$$\lambda \left[x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - y \right] = C_1 \lambda^3 x^3 + C_2 \lambda^5 x^2 y (y^2 - x^2)$$

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - y = C_1 \lambda^2 x^3 + C_2 \lambda^4 x^2 y (y^2 - x^2)$$

Let $C_1 \lambda^2 = 1, \quad \lambda = \frac{1}{\sqrt{C_1}}$

The equation becomes

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - y = x^3 + \left(\frac{C_2}{C_1^2} \right) x^2 y (y^2 - x^2)$$

$$\frac{C_2}{C_1^2} = \frac{1}{K_2} \cdot \frac{K_1^2}{K_1^2} = \frac{K_2}{K_1^2} = \frac{1}{6\beta^4} \left(\frac{t}{r} \right)^2 \frac{E^2 \beta^4}{4\sigma^2} = \frac{1}{24} \frac{1}{\left[\frac{\sigma}{E \left(\frac{t}{r} \right)} \right]^2}$$

$$= \frac{1}{24} \frac{1}{\phi^2}$$

where $\phi = \frac{\sigma}{E \left(\frac{t}{r} \right)}$

$$x = \frac{\alpha}{\beta}$$

$$x = \frac{\alpha}{\lambda} = \frac{\alpha_r}{\beta} \cdot \sqrt{C_1} = \frac{\alpha_r}{\beta} \sqrt{\frac{K_1}{K_2}}$$

$$= \frac{\alpha_r}{\beta} \sqrt{\frac{2\sigma}{E\beta^2} \cdot \frac{6\beta^4}{\left(\frac{t}{r} \right)^2}} = \frac{2\alpha_r}{\left(\frac{t}{r} \right)} \sqrt{\frac{3\sigma}{E}}$$

$$= \frac{2\sqrt{3}\alpha_r}{\left(\frac{t}{r} \right)^{\frac{1}{2}}} \phi$$

$$y = \frac{2\sqrt{3}\Theta_r}{\left(\frac{t}{r} \right)^{\frac{1}{2}}} \phi$$

α_r, Θ_r , thermal variables

$$x = \frac{1}{\sqrt{2\Delta \frac{1}{f}}} \alpha, \quad y = \frac{1}{\sqrt{2\Delta \frac{1}{f}}} \phi,$$

where $\Delta = \frac{1}{24\phi^2}$.

$$\left[x^3 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - y \left\{ 1 + \Delta x^2 (y^2 - x^2) \right\} = x^2 \right]$$

For corresponding variation problem we write the differential equation as

$$x \frac{d^2 y}{dx^2} + \frac{dy}{dx} - \frac{y}{x} - \Delta x y (y^2 - x^2) - x^2 = 0$$

$$I = \frac{\Delta}{2} \int_0^x (y^2 - x^2)^2 x dx + \int_0^x \left[\left(\frac{dy}{dx} - 1 \right)^2 + \left(\frac{y}{x} - 1 \right)^2 \right] x dx + 2 \int_0^x x^3 y dx$$

To transform the differential equation into integral equation

$$\frac{d}{dx} \left[x \frac{dy}{dx} \right] = \left\{ \frac{y}{x} + \Delta x y (y^2 - x^2) \right\} + x^2$$

$$x \frac{dy}{dx} = \int_0^x \left\{ \frac{y}{\xi} + \Delta \xi y (y^2 - \xi^2) \right\} d\xi + \frac{x^3}{3} + C_1$$

When $\frac{dy}{dx} = 0$ at $x=0$, $C_1 = 0$

$$\frac{dy}{dx} = \frac{1}{x} \int_0^x \left\{ \frac{y}{\xi} + \Delta \xi y (y^2 - \xi^2) \right\} d\xi + \frac{x^2}{3} + \frac{C_1}{x}$$

$$y = \int_0^x \frac{dt}{t} \int_0^t \left\{ \frac{y}{\xi} + \Delta \xi y (y^2 - \xi^2) \right\} d\xi + \frac{x^3}{9} + C_1 \log x + C_2$$

When $x=x_0$, $y=y_0$; $x=0$, $y=0$

42)

$$y = \int_0^x \log\left(\frac{x}{t}\right) \left\{ \frac{t}{x} + \Delta t y (y^2 - t^2) \right\} dt + \frac{x^3}{9} + C_1 \log x + C_2$$

$$x_0 = \int_0^{x_0} \log\left(\frac{x_0}{t}\right) \left\{ \frac{t}{x_0} + \Delta t y (y^2 - t^2) \right\} dt + \frac{x_0^3}{9} + C_1 \log x_0 + C_2$$

$$0 = C_1 (-\infty) + C_2$$

$$\therefore C_1 = 0, C_2 = 0.$$

$$\therefore \boxed{y = \int_0^x \log\left(\frac{x}{t}\right) \left\{ \frac{t}{x} + \Delta t y (y^2 - t^2) \right\} dt + \frac{x^3}{9}}$$

This is the integral equation for "y" of second kind.

The 2nd energy

$$\frac{E\left(\frac{1}{r}\right)}{4} \int_0^{\beta} \int_0^1 (\Theta^2 - \alpha^2)^2 d\alpha + \frac{E\left(\frac{1}{r}\right)}{12} \beta^2 \int_0^1 \left\{ \left(\frac{d\Theta}{d\alpha} - 1 \right)^2 + \left(\frac{\Theta}{\alpha} - 1 \right)^2 \right\} \alpha d\alpha$$

$$+ \frac{1}{4} \beta^4 \int_0^1 x^2 \Theta d\alpha$$

$$\frac{1}{4} \beta^2 \int_0^1 (\Theta^2 - \alpha^2)^2 d\alpha + \frac{\left(\frac{1}{r}\right)^2}{12} \frac{1}{\beta^2} \int_0^1 \left\{ \left(\frac{d\Theta}{d\alpha} - 1 \right)^2 + \left(\frac{\Theta}{\alpha} - 1 \right)^2 \right\} \alpha d\alpha + \left(\frac{2\pi}{E} \right) \int_0^1 x^2 \Theta d\alpha$$

$$\beta^2 = \frac{\left(\frac{1}{r}\right)}{\sqrt{3}} \left[\frac{\int_0^1 \left\{ \left(\frac{d\Theta}{d\alpha} - 1 \right)^2 + \left(\frac{\Theta}{\alpha} - 1 \right)^2 \right\} \alpha d\alpha}{\int_0^1 (\Theta^2 - \alpha^2)^2 d\alpha} \right]^{\frac{1}{2}}$$

Put $y = w + x$

$y = w + x$

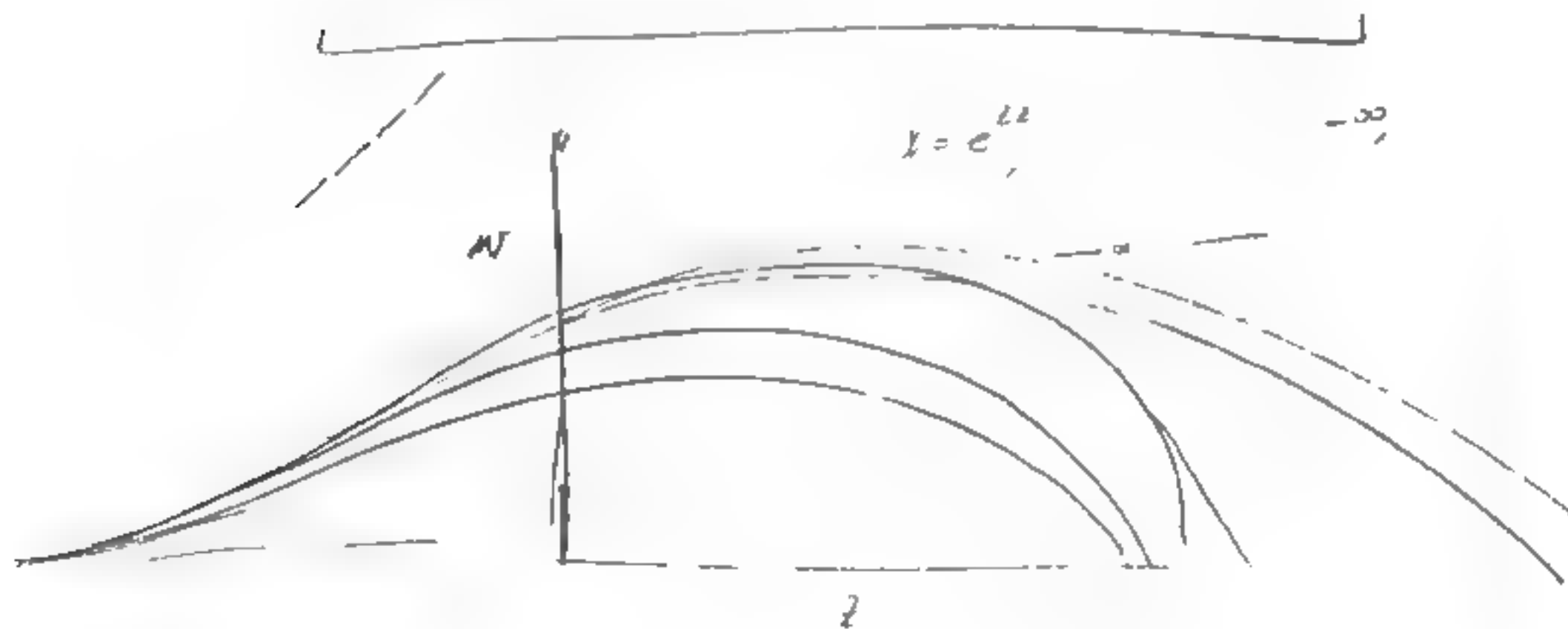
$y^2 = w^2 + 2wx + x^2$

$$\frac{dy}{dx} = \frac{dw}{dx} + 1, \quad \frac{d^2y}{dx^2} = \frac{d^2w}{dx^2}$$

$$x^2 \frac{d^2w}{dx^2} + x \frac{dw}{dx} + x - (w+x) \left\{ 1 + \frac{1}{2} x^2 (w^2 + 2wx) \right\} = x^3$$

$$x^2 \frac{d^2w}{dx^2} + x \frac{dw}{dx} - w \left\{ 1 + \frac{1}{2} x^2 (w+2x)(w+x) \right\} = x^3$$

$$\text{Let } \frac{d^2w}{dx^2} - w \left\{ 1 + \frac{1}{2} e^{2x} (w+2e^x)(w+e^x) \right\} = e^{3x}$$



$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - y \{1 + \Delta x^2 (y^2 - x^2)\} = x^3$$

2/3)
a

Put $v = y - x$ $y = v + x$

$$x^2 \frac{d^2 v}{dx^2} + x \frac{dv}{dx} - v \{1 + \Delta x^2 (v+x)(v+2x)\} = x^3$$

Put $w = x^{\frac{1}{2}} v$, $v = \frac{w}{x^{\frac{1}{2}}}$

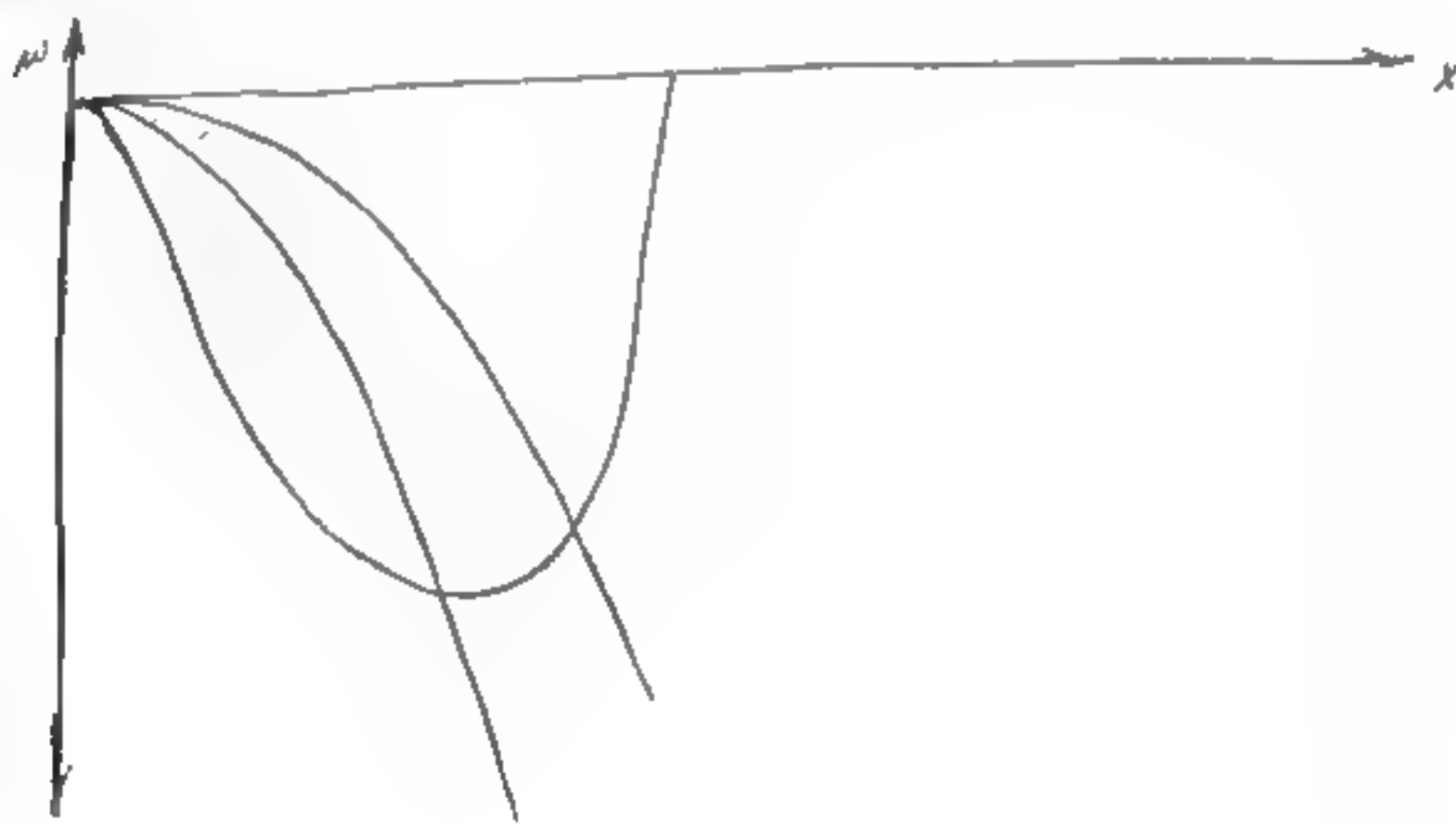
$$\frac{dv}{dx} = \frac{1}{x^{\frac{1}{2}}} \frac{dw}{dx} - \frac{1}{2} \frac{1}{x^{\frac{3}{2}}} w$$

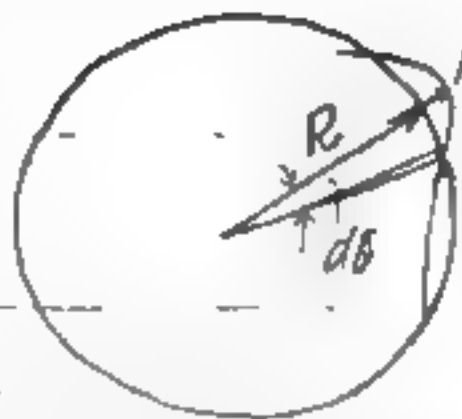
$$\frac{d^2 v}{dx^2} = \frac{1}{x^{\frac{1}{2}}} \frac{d^2 w}{dx^2} - \frac{1}{x^{\frac{3}{2}}} \frac{dw}{dx} + \frac{3}{4} \frac{1}{x^{\frac{5}{2}}} w$$

$$\therefore x^{\frac{3}{2}} \frac{d^2 w}{dx^2} - x^{\frac{1}{2}} \frac{dw}{dx} + \frac{3}{4} \frac{1}{x^{\frac{1}{2}}} w + x^{\frac{1}{2}} \frac{dw}{dx} - \frac{1}{2} \frac{1}{x^{\frac{1}{2}}} w$$

$$= \frac{w}{x^{\frac{1}{2}}} - \Delta \frac{w}{x^{\frac{1}{2}}} x^2 \left(\frac{w}{x^{\frac{1}{2}}} + x \right) \left(\frac{w}{x^{\frac{1}{2}}} + 2x \right) = x^3$$

$$\frac{d^2 w}{dx^2} = \frac{3}{4} \frac{w}{x^{\frac{1}{2}}} + \Delta \frac{w}{x} (w + x^{\frac{3}{2}})(w + 2x^{\frac{3}{2}}) + x^{\frac{3}{2}}$$





$$(ds)_0^2 = R^2 d\theta^2$$

$$(ds)^2 = (dr)^2 + r^2 d\psi^2$$

$$= db^2 \left(\frac{dw}{db} \right)^2 + (R-w)^2 \left[1 + \frac{1}{R} \frac{du}{d\theta} \right]^2 d\theta^2$$

$$r = R - w$$

$$\psi = \theta + \frac{u}{R}$$

$$\epsilon = \frac{db \left[\sqrt{\left(\frac{dw}{db} \right)^2 + (R-w)^2 \left[1 + \frac{1}{R} \frac{du}{d\theta} \right]^2} - R \right]}{R db}$$

$$= \sqrt{\left(\frac{1}{R} \frac{dw}{d\theta} \right)^2 + \left(1 + \frac{w}{R} \right)^2 \left(1 + \frac{1}{R} \frac{du}{d\theta} \right)^2} - 1$$

$$= \sqrt{1 - 2\left(\frac{w}{R}\right) + \left(\frac{1}{R} \frac{dw}{d\theta}\right)^2 + \left(1 + \frac{w}{R}\right)^2 \left(1 + \frac{1}{R} \frac{du}{d\theta}\right)^2} - 1$$

$$= -\left(\frac{w}{R}\right) + \frac{1}{2} \frac{dw}{d\theta} + \frac{1}{2} \left(\frac{1}{R} \frac{dw}{d\theta} \right)^2 = \frac{1}{R} \frac{du}{d\theta} - \frac{w}{R} + \frac{1}{2} \left(\frac{1}{R} \frac{dw}{d\theta} \right)^2 = \epsilon$$

The total strain energy

$$E_s = \frac{EI}{2} \int \left(\frac{d^2 v}{db^2} + \frac{du}{d\theta} \right)^2 \frac{1}{R^3} R db + \frac{EA}{2} \int \left[\frac{1}{R} \frac{dw}{d\theta} - \frac{w}{R} + \frac{1}{2} \left(\frac{1}{R} \frac{dw}{d\theta} \right)^2 \right]^2 R db$$

where I moment of inertia & A area

$$\begin{aligned} E_s &= \frac{E}{2} \left\{ \frac{1}{R} \int \left(\frac{1}{R} \frac{dw}{d\theta} + \frac{du}{d\theta} \right)^2 d\theta + AR \int \left[\frac{1}{R} \frac{dw}{d\theta} - \frac{w}{R} + \frac{1}{2} \left(\frac{1}{R} \frac{dw}{d\theta} \right)^2 \right]^2 d\theta \right\} \\ &= \frac{EAR}{2} \left\{ \left(\frac{1}{R} \right)^2 \int \left(\frac{1}{R} \frac{dw}{d\theta} + \frac{du}{d\theta} \right)^2 d\theta + \int \left[\frac{1}{R} \frac{dw}{d\theta} - \frac{w}{R} + \frac{1}{2} \left(\frac{1}{R} \frac{dw}{d\theta} \right)^2 \right]^2 d\theta \right\} \end{aligned}$$

If $u=0$,

$$E_s = \frac{EAR}{2} \left\{ \left(\frac{u}{R} \right)^2 \int \left[\frac{1}{R} \frac{d^2 w}{d\theta^2} \right]^2 d\theta + \int \left[\frac{1}{2} \left(\frac{1}{R} \frac{dw}{d\theta} \right)^2 - \frac{w^2}{R} \right] d\theta \right\}$$

For frame

For stiffener

$$E_s = \frac{EI_s}{2} \int \left(\frac{dw}{dx^2} \right)^2 dx$$

For frame: Putting $\frac{w}{R} = \beta [\cos(n\theta) + \cos(2n\theta)]$

$$\frac{1}{R} \frac{dw}{d\theta} = \beta [-n \sin n\theta - 2n \sin(2n\theta)]$$

$$\frac{1}{R} \frac{d^2 w}{d\theta^2} = \beta [-n^2 \cos n\theta - 4n^2 \cos(2n\theta)]$$

$$= -\beta n^2 [\cos n\theta + 4 \cos(2n\theta)]$$

$$2 \int_0^{\pi/n} \left[\frac{1}{R} \frac{d^2 w}{d\theta^2} \right]^2 d\theta = 2 \beta^2 n^4 \int_0^{\pi} (\cos \psi + 4 \cos 2\psi)^2 d\psi$$

$$= \pi \beta^2 n^4 \cdot 17 = \underline{\underline{17\pi \beta^2 n^4}}$$

$$\left(\frac{1}{R} \frac{dw}{d\theta} \right)^2 = \beta^2 n^2 [\sin n\theta + 2 \sin(2n\theta)]^2$$

$$= \beta^2 n^2 [\sin^2 n\theta + 4 \sin n\theta \sin(2n\theta) + 4 \sin^2(2n\theta)]$$

$$\begin{aligned} \left(\frac{1}{R} \frac{dw}{d\theta}\right)^2 &= \beta^2 n^2 \left[\frac{1}{2}(1 - \cos 2n\theta) + 2(\cos n\theta - \cos 3n\theta) + 2(1 - \cos 4n\theta) \right] \\ &= \beta^2 n^2 \left[\frac{5}{2} + 2 \cos n\theta - \frac{1}{2} \cos 2n\theta - 2 \cos 3n\theta - 2 \cos 4n\theta \right] \end{aligned}$$

$$\begin{aligned} \frac{1}{2} \left(\frac{1}{R} \frac{dw}{d\theta}\right)^2 - \frac{w^2}{R} &= \beta^2 n^2 \left[\frac{5}{4} + \cos n\theta - \frac{1}{4} \cos 2n\theta - \cos 3n\theta - \cos 4n\theta \right] \\ &\quad - \beta [\cos n\theta + \cos 2n\theta] \end{aligned}$$

$$= \beta \left[\frac{5}{4} \beta n^2 + (\beta n^2 - 1) \cos n\theta - \left(\frac{\beta n^2}{4} + 1 \right) \cos 2n\theta - \beta n^2 \cos 3n\theta - \beta n^2 \cos 4n\theta \right]$$

$$2 \int_0^{\frac{\pi}{n}} \left[\frac{1}{2} \left(\frac{1}{R} \frac{dw}{d\theta}\right)^2 - \frac{w^2}{R} \right] d\theta$$

$$= \frac{\pi \beta^2}{n} \left[\frac{25}{8} (\beta n^2)^2 + (\beta n^2 - 1)^2 + \left(\frac{\beta n^2}{4} + 1 \right)^2 + 2 (\beta n^2)^2 \right]$$

$$= \frac{\pi \beta^2}{n} \left[\frac{99}{16} (\beta n^2)^2 - \frac{3}{2} (\beta n^2) + 2 \right]$$

$$E_s = \frac{E A R}{2} \left\{ \left(\frac{w}{R}\right)^2 17 \pi \beta^2 n^3 + \frac{\pi \beta^2}{n} \left[\frac{99}{16} (\beta n^2)^2 - \frac{3}{2} (\beta n^2) + 2 \right] \right\}$$

$$= \frac{E A R \pi \beta^2}{2} \left\{ \left(\frac{w}{R}\right)^2 17 n^3 + \frac{1}{n} \left[\frac{99}{16} (\beta n^2)^2 - \frac{3}{2} \beta n^2 + 2 \right] \right\}$$

$$= \frac{E A R \pi}{2} \left\{ \beta^2 \left[\left(\frac{w}{R}\right)^2 17 n^3 + \frac{2}{n} \right] + \beta^4 \frac{99}{16} n^3 - \beta^3 \frac{3}{2} n \right\}$$

$$E_{\text{total}} = \frac{E A R \pi}{2} \left\{ \frac{1}{R^2} N_1 a^2 \left[\left(\frac{w}{R}\right)^2 17 n^3 + \frac{2}{n} \right] + \frac{1}{R^4} N_3 a^4 \frac{99}{16} n^3 - \frac{1}{R^3} N_2 a^2 n \right\}$$

$$\mathcal{E}_{\text{tot},s} = 2\pi^5 \left(\frac{1}{n}\right) a^2 \left(\frac{R}{d}\right) \left(\frac{E I_s}{L^3}\right)$$

$$I_m = \frac{1}{2} \pi^2 \left(\frac{P}{d}\right) \left(\frac{R}{L}\right) a^2 (M_1 + R_1)$$

Therefore the total potential energy of the system

$$\mathcal{E} = \frac{E A_f R \pi}{2} \left\{ \frac{1}{R^2} N_1 a^2 \left[\left(\frac{L}{R}\right)^2 17 n^3 + \frac{2}{n} \right] - \frac{1}{R^3} N_2 a^3 \frac{3}{2} n + \frac{1}{R^4} N_3 a^4 \frac{99}{16} n^3 \right\} \\ + 2\pi^5 \left(\frac{1}{n}\right) a^2 \left(\frac{R}{d}\right) \frac{E I_s}{L^3} - \frac{1}{2} \pi^2 \left(\frac{P}{d}\right) \left(\frac{R}{L}\right) a^2 (M_1 + R_1)$$

$$1) \quad \frac{\partial \mathcal{E}}{\partial a} = 0 \quad \frac{\partial \mathcal{E}}{\partial n} = 0 \quad \text{The first condition gives}$$

$$\frac{E A_f R \pi}{2} \left\{ \frac{N_1}{R^2} \left[\left(\frac{L}{R}\right)^2 17 n^2 + \frac{2}{n^2} \right] - \frac{15}{R^3} N_2 a n + \frac{N_3}{R^4} a^2 \frac{99}{8} n^2 \right\} \\ + 4\pi^5 \left(\frac{1}{n}\right) \left(\frac{R}{d}\right) \frac{E I_s}{L^3} = \pi^2 \left(\frac{P}{d}\right) \left(\frac{R}{L}\right) (M_1 + R_1)$$

$$\frac{(-1)(-2)}{2!}$$

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Assuming $n=1$

$$EA_f R \left\{ \frac{N_1}{R^2} \left[17 \left(\frac{L}{R} \right)^2 + 2 \right] - 2.25 \frac{N_2}{R^3} a + 12.375 \frac{N_3}{R^4} a^2 \right\}$$

$$+ 4\pi^2 \left(\frac{R}{a} \right) \frac{EI_s}{L^3} = \pi \left(\frac{P}{a} \right) \left(\frac{R}{L} \right) (M_1 + R_1)$$

or $2.25 N_2 = 24.75 N_3 / R \cdot a$

$$\therefore EA_f \frac{1}{R} \left\{ N_1 \left[17 \left(\frac{L}{R} \right)^2 + 2 \right] - 0.1022 \frac{N_2^2}{N_3} \right\} + 4\pi^2 \left(\frac{R}{a} \right) \frac{EI_s}{L^3} = \pi \left(\frac{P}{a} \right) \left(\frac{R}{L} \right) (M_1 + R_1)$$

$$M_1 = \sin\left(\frac{\pi}{n}\right) \left[1 - \frac{1}{1-n^2} + \frac{\frac{1}{2}}{1-4n^2} - \frac{1}{1-9n^2} + \frac{\frac{1}{2}}{1-16n^2} \right]$$

$$= \sin\left(\frac{\pi}{n}\right) \left(-\frac{1}{1-n^2} \right) \quad \text{when } n=1$$

$$= - \frac{\sin \frac{\pi}{n}}{1-n^2} = - \frac{\cos \frac{\pi}{n} \cdot \left(-\frac{\pi}{n^2} \right)}{-2n} = \frac{\pi}{2}$$

$$R_1 = \frac{\pi}{4}$$

$$M_1 + R_1 = \frac{3\pi}{4}$$

$$\frac{E}{R} \left\{ N_1 \left[17 \frac{L}{R^2} + 2 A_f \right] - 0.1022 \frac{N_2^2}{N_3} A_f \right\} + E \pi^4 \left(\frac{R}{d} \right) \frac{L^3}{L^3}$$

$$= \pi^4 \frac{3}{4} \left(\frac{P}{d} \right) \left(\frac{R}{L} \right)$$

$$4N_1 = \sum_{k=0}^n \left[1 - \cos \left(\frac{2\pi k}{n} \right) \right]^2$$

$$= (n+2) \sum_{k=0}^n \cos \left(\frac{2\pi k}{n} \right) + \sum_{k=0}^n \cos^2 \left(\frac{2\pi k}{n} \right)$$

$$= (n+2) \sum_{k=0}^n \cos \left(\frac{2\pi k}{n} \right) + \frac{1}{2} \sum_{k=0}^n \left(1 + \cos \frac{4\pi k}{n} \right)$$

$$= \frac{3n+1}{2} - 2 \sum_{k=0}^n \cos \left(\frac{2\pi k}{n} \right) + \frac{1}{2} \sum_{k=0}^n \cos \frac{4\pi k}{n}$$

$$= \frac{3n}{2} - 2 \sum_{k=0}^n \cos \frac{2\pi k}{n} = \frac{3n}{2} - 2 \sum_{k=1}^n \cos \frac{2\pi k}{n} \quad t = \frac{2\pi}{n}$$

$$2 \sum_{k=0}^n \cos kb = \sum_{k=0}^n e^{ibk} + \sum_{k=0}^n e^{-ibk} = \sum_{k=0}^n (e^{ib})^k + \sum_{k=0}^n (e^{-ib})^k$$

$$(e^{ib}) \sum_{k=0}^n (e^{ib})^k - \sum_{k=0}^n (e^{ib})^k = (e^{ib})^{n+1} - 1$$

$$\therefore \sum_{k=0}^n (e^{ib})^k = \frac{(e^{ib})^{n+1} - 1}{e^{ib} - 1}$$

$$\begin{aligned}
\sum_{k=0}^n (e^{i\theta})^k + \sum_{k=0}^n (e^{-i\theta})^k &= \frac{(e^{i\theta})^{n+1} - 1}{e^{i\theta} - 1} + \frac{(e^{-i\theta})^{n+1} - 1}{e^{-i\theta} - 1} \\
&= \frac{1 - e^{i(n+1)\theta} + e^{i(n+1)\theta} - e^{i(n+1)\theta} - 1}{1 - e^{i\theta}} \\
&= 1 + \frac{e^{i(n+1)\theta} - e^{i(n+1)\theta} - 1}{1 - e^{i\theta}} = 1 + \frac{1 - e^{i\frac{2\pi}{n}}}{1 - e^{i\frac{2\pi}{n}}} \\
&= 2
\end{aligned}$$

$$\therefore 4N_1 = \frac{3(n+1)}{2} - 2 + \frac{1}{2} = \frac{3(n+1)}{2} - \frac{3}{2} = \frac{3}{2}[(n+1) - 1]$$

$$\underline{\underline{N_1 = \frac{3}{8}n \quad \text{for } n > 2}}$$

$$\begin{aligned}
8N_2 &= \sum_{k=0}^n \left[1 - \cos\left(\frac{2\pi k}{n}\right)\right]^3 \\
&= (n+1) - 3 \sum_{k=0}^n \cos \frac{2\pi k}{n} + 3 \sum_{k=0}^n \cos^2 \frac{2\pi k}{n} - \sum_{k=0}^n \cos^3 \frac{2\pi k}{n} \\
&= \frac{5}{2}(n+1) - \frac{15}{2} + 3 - \frac{1}{2} = \frac{5}{2}[(n+1) - 2] = \frac{5}{2}(n-1)
\end{aligned}$$

$$\underline{\underline{N_2 = \frac{5}{16}n \quad \text{for } n > 2}}$$

$$\begin{aligned}
16N_3 &= \sum_{k=0}^n \left[1 - \cos\left(\frac{2\pi k}{n}\right)\right]^4 \\
&= \sum_{k=0}^n \left[1 - 4 \cos\left(\frac{2\pi k}{n}\right) + 6 \cos^2\left(\frac{2\pi k}{n}\right) - 4 \cos^3\left(\frac{2\pi k}{n}\right) + \cos^4\left(\frac{2\pi k}{n}\right)\right] \\
&= \frac{35}{8}(n+1) - 14 + 3 - 2 + \frac{1}{4} = \frac{35}{8}n
\end{aligned}$$

$$N_3 = \frac{35\pi}{128}$$

$$\frac{E}{R} \left\{ \frac{3}{8}(n+1) \left[17 \frac{I_f}{R^2} + 2A_f \right] - 0.1022 \frac{\frac{25}{16^2}(n+1)}{\frac{35}{16 \times 8}} A_f \right\}$$

$$\therefore \frac{3\pi^2}{4} \left(\frac{P}{d} \right) \left(\frac{R}{L} \right) = E \left[\frac{1}{R} \left\{ \frac{3}{8}(n+1) \left(17 \frac{I_f}{R^2} + 2A_f \right) - 0.1022 \times \frac{5}{14} (n+1) A_f \right\} + \pi^4 \left(\frac{R}{d} \right) \frac{I_s}{L^3} \right]$$

$$\begin{aligned} \text{Moment} &= 4 \int_0^{\frac{\pi}{2}} dP \cdot R \sin \phi = 4 R^2 \left(\frac{P}{d} \right) \int_0^{\frac{\pi}{2}} \cos^2 \phi d\phi \\ &= \pi R^2 \left(\frac{P}{d} \right) = M \end{aligned}$$

$$\begin{aligned} \frac{3}{4} \pi \left(\frac{M}{R^2} \right) \left(\frac{R}{L} \right) &= E(n+1) \left\{ \frac{1}{R} \left[\frac{3}{8} \left(17 \frac{I_f}{R^2} + 2A_f \right) - 0.1022 \times \frac{5}{14} A_f \right] \right. \\ &\quad \left. + E \pi^4 \left(\frac{R}{d} \right) \frac{I_s}{L^3} \right\} \end{aligned}$$

$$\begin{aligned} M &= \frac{4}{3\pi} E R L \left[\frac{(n+1)}{R} \left\{ \frac{3}{8} \left(17 \frac{I_f}{R^2} + 2A_f \right) - 0.1022 \times \frac{5}{14} A_f \right\} \right. \\ &\quad \left. + \pi^4 \left(\frac{R}{d} \right) \frac{I_s}{L^3} \right] \end{aligned}$$

$$\begin{aligned} m &= 31, & R &= 75.76'' & L &= 64'' & A_f &= 0.0291 \\ & & d &= 2.53'' & & & I_f &= 0.00001532 \\ & & & & & & I_s &= 0.000374 \end{aligned}$$

$$\begin{aligned}
 & \frac{n+1}{R} \left\{ \frac{3}{8} \left(17 \frac{I_f}{R^2} + 2 A_f \right) - 0.1022 \times \frac{5}{14} A_f \right\} \\
 &= \frac{32}{15.76} \left\{ 0.375 \left(17 \times \frac{0.00001537}{15.76^2} + 2 \times 0.0291 \right) - 0.1022 \times \frac{5}{14} \times 0.0291 \right\} \\
 &= \frac{32}{15.76} \left\{ 0.375 (0.000001051 + 0.0582) - 0.001062 \right\}
 \end{aligned}$$

$$= \frac{32}{15.76} \times 0.02074 = 0.0421$$

$$\pi^* \left(\frac{15.26}{2.53} \right) \frac{0.000374}{64^3} = \frac{0.0000554}{64}$$

$$M = \frac{1.3333}{\pi} \times 10.6 \times 10^6 \times 15.76 \times 64 \times 0.0421 \quad \text{Too big}$$

Approximate Calculation

259

Assuming equipartition of energy in frame

Putting $\left(\frac{i}{R}\right) \left[\frac{1}{R} \frac{d^2 w}{dt^2} + \frac{1}{R} \frac{dw}{dt} \right] = \frac{1}{R} \frac{dw}{dt} - \frac{w}{R} + \frac{1}{2} \left(\frac{1}{R} \frac{dw}{dt} \right)^2$

$$\left(1 - \frac{i}{R}\right) \frac{1}{R} \frac{dw}{dt} = \frac{i}{R} \frac{1}{R} \frac{d^2 w}{dt^2} + \frac{w}{R} - \frac{1}{2} \left(\frac{1}{R} \frac{dw}{dt} \right)^2$$

$$\frac{1}{R} \frac{dw}{dt} = \frac{1}{1 - \left(\frac{i}{R}\right)} \left[\frac{i}{R} \frac{1}{R} \frac{d^2 w}{dt^2} + \frac{w}{R} - \frac{1}{2} \left(\frac{1}{R} \frac{dw}{dt} \right)^2 \right]$$

$$\frac{1}{R} \frac{d^2 w}{dt^2} + \frac{1}{R} \frac{dw}{dt} = \frac{1}{1 - \left(\frac{i}{R}\right)} \left\{ \frac{1}{R} \frac{d^2 w}{dt^2} + \frac{w}{R} - \frac{1}{2} \left(\frac{1}{R} \frac{dw}{dt} \right)^2 \right\}$$

$$\approx \underline{\underline{\frac{1}{R} \frac{d^2 w}{dt^2} + \frac{w}{R} - \frac{1}{2} \left(\frac{1}{R} \frac{dw}{dt} \right)^2}}$$

For the assumed buckling form

$$\frac{1}{R} \frac{d^2 w}{dt^2} + \frac{w}{R} - \frac{1}{2} \left(\frac{1}{R} \frac{dw}{dt} \right)^2$$

$$= -\beta \left\{ \left[\pi^2 \cos \pi b + 4\pi^2 \cos(2\pi b) \right] + \frac{5}{4} \beta \pi^2 + (\beta \pi^2 - 1) \cos \pi b \right. \\ \left. - \left(\frac{\beta \pi^2}{4} + 1 \right) \cos 2\pi b - \beta \pi^2 \cos 3\pi b - \beta \pi^2 \cos 4\pi b \right\}$$

$$= -\beta \left\{ \frac{5}{4} \beta \pi^2 + (\beta \pi^2 + \pi^2 - 1) \cos \pi b + (4\pi^2 - \frac{\beta \pi^2}{4} - 1) \cos 2\pi b \right. \\ \left. - \beta \pi^2 \cos 3\pi b + \beta \pi^2 \cos 4\pi b \right\}$$

$$\int \left[\frac{1}{R} \frac{d^2 w}{dt^2} + \frac{w}{R} - \frac{1}{2} \left(\frac{1}{R} \frac{dw}{dt} \right)^2 \right]^2 dt$$

$$= \frac{\pi \beta^2}{n} \left\{ \frac{25}{8} (\beta \pi^2)^2 + (\beta \pi^2 + \pi^2 - 1)^2 + (4\pi^2 - \frac{\beta \pi^2}{4} - 1)^2 + 2 (\beta \pi^2)^2 \right\}$$

$$\begin{aligned}
&= \frac{\pi \beta^2}{n} \left\{ \frac{25}{8} (\beta n^2)^2 + (\beta n^2)^2 + n^4 + 1 + 2\beta n^4 - 2\beta n^2 - 2n^2 \right. \\
&\quad \left. + 16n^4 - 2\beta n^4 - 8n^2 + \frac{\beta n^2}{2} + 1 + 2(\beta n^2)^2 + \frac{(\beta n^2)^2}{16} \right\} \\
&= \frac{\pi \beta^2}{n} \left\{ \frac{99}{16} (\beta n^2)^2 - \frac{3}{2} \beta n^2 + 2 + 17n^4 - 10n^2 \right\} \\
&= \frac{\pi}{n} \left\{ \frac{99}{16} n^4 \beta^4 - \frac{3}{2} n^2 \beta^3 + (2 + 17n^4 - 10n^2) \beta^2 \right\}
\end{aligned}$$

$$E_{totf} = \frac{E I_f}{2 R^3} \frac{\pi}{n} \left\{ \frac{99}{16} n^4 N_3 \frac{a^4}{R^2} - 1.5 n^2 N_2 \frac{a^3}{R} + (2 + 17n^4 - 10n^2) N_1 a^2 \right\}$$

When $n=1$

$$E_{totf} = \frac{E I_f}{2 R^3} \pi \left\{ \frac{99}{16} N_3 \frac{a^4}{R^2} - 1.5 N_2 \frac{a^3}{R} + 9 N_1 a^2 \right\}$$

$$\frac{E I_f}{R^3} \left\{ \frac{99}{8} N_3 \frac{a^2}{R^2} - 2.25 N_2 \frac{a}{R} + 9 N_1 \right\} + 4 \pi^4 \left(\frac{R}{d} \right) \frac{E I_s}{L^3}$$

$$= \pi \left(\frac{P}{d} \right) \left(\frac{R}{L} \right) (M_1 + R_1)$$

$$\frac{99}{4} N_3 \frac{a}{R} = 2.25 N_2 \quad / \quad a = \frac{1}{11} \frac{N_2}{N_3} R$$

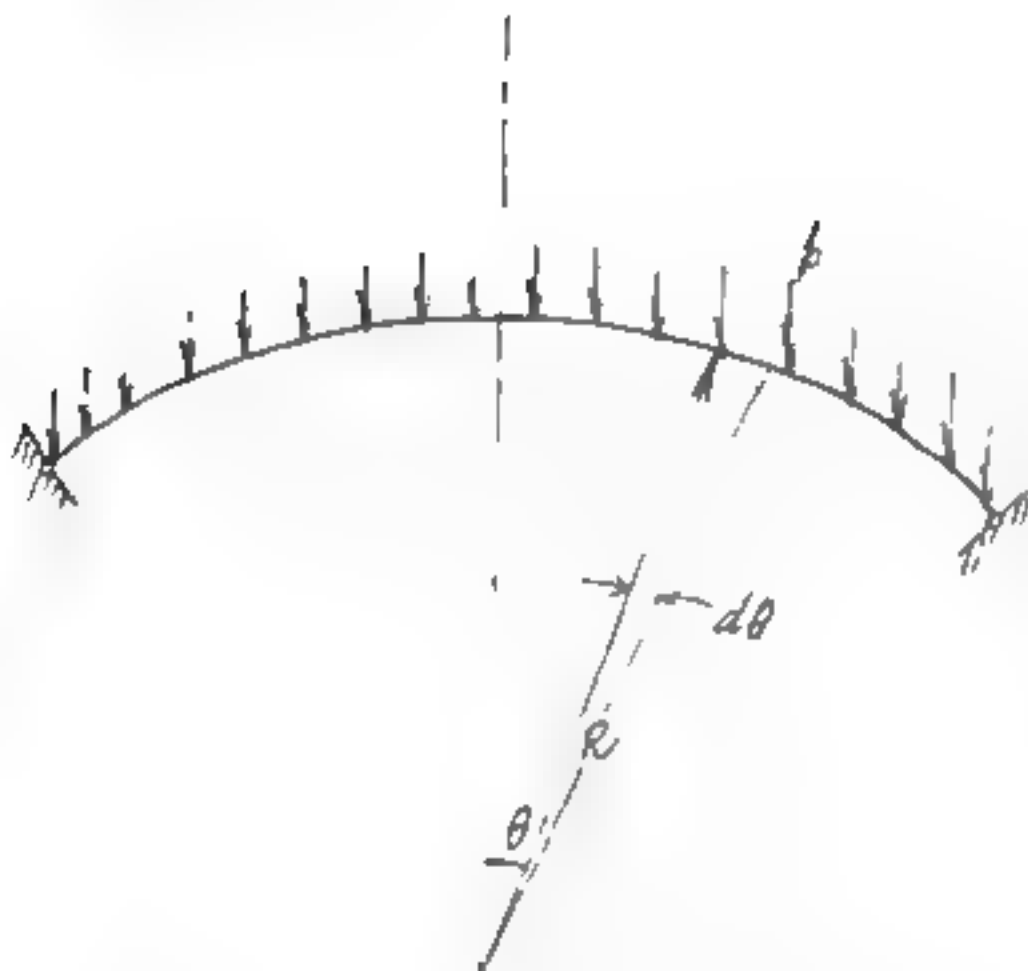
$$\frac{E I_f}{R^3} \left\{ 9 N_1 - \frac{9}{88} \frac{N_2^2}{N_3} \right\} + 4 \pi^4 \left(\frac{R}{d} \right) \frac{E I_s}{L^3} = \pi \left(\frac{P}{d} \right) \left(\frac{R}{L} \right) (M_1 + R_1)$$

Approximately, $\frac{3\pi^2}{4} \left(\frac{P}{d}\right) \left(\frac{L}{L}\right) = E \left[\frac{I_f}{R^3} (m+1) \left(\frac{27}{8} - \frac{9}{88} \frac{5}{14} \right) + 4\pi^2 \left(\frac{P}{d}\right) \frac{Z_s}{L^3} \right]$

$$M_6 = \frac{4}{3\pi} E R L \left\{ (m+1) \frac{I_f}{R^3} 3.347 + 4\pi^2 \frac{P}{d} \frac{Z_s}{L^3} \right\}$$

$$(m+1) \frac{I_f}{R^3} 3.347 = \frac{3.347}{R} \frac{0.00001537}{1576^2} \times 32$$

$$= \frac{0.0000043}{R} \quad \text{Too small}$$



1)
The original form of the shell is spherical

i.e. suppose the deflected form of the shell is axially symmetrical

$$\theta_1 = \theta_0 + \theta_0 f(\theta_0) = \theta_0 [1 + f(\theta_0)]$$

$$R = R_0 + R g(\theta_0)$$

$$= R [1 + g(\theta_0)]$$

The original length of the element $(ds)_0 = R(d\theta)_0$

The new length of the element

$$= \sqrt{R^2(d\theta)^2 + (dR)^2}$$

$$= \sqrt{R^2[1 + g(\theta_0)]^2 \left[(1 + f(\theta_0)) d\theta_0 + \theta_0 f'(\theta_0) d\theta_0 \right]^2 + R^2[g'(\theta_0)]^2 (d\theta_0)^2}$$

$$= R \sqrt{R^2 [1 + g(\theta_0)]^2 [1 + f(\theta_0) + \theta_0 f'(\theta_0)]^2 + [g'(\theta_0)]^2} d\theta_0$$

If the deflection is inextensional, in the sense that

$$(ds)_0 = ds,$$

Then
$$\frac{[1 + g(\theta_0)]^2 [1 + f(\theta_0) + \theta_0 f'(\theta_0)]^2 + [g'(\theta_0)]^2 - 1}{R^2} = 0$$



The distance of the element from the axis is

$$R \sin \theta_0$$

2)

$$\left(\frac{1}{2} E \epsilon^2 V \right)$$

before deflection.

The distance is $R \sin \theta$, after deflection.

$$R [1 + f(\theta_0)] \sin \{ \theta_0 [1 + f(\theta_0)] \}$$

$$\sin \{ \theta_0 + \theta_0 f(\theta_0) \}$$

The change in length of the ring ds is

$$2\pi R \left[[1 + f(\theta_0)] \sin \{ \theta_0 [1 + f(\theta_0)] \} - \sin \theta_0 \right]$$

The strain energy stored in this ds is

$$\frac{1}{2} [E \epsilon] t ds \quad t = \text{thickness}$$

Now

$$\epsilon = [1 + f(\theta_0)] \frac{\sin \{ \theta_0 [1 + f(\theta_0)] \}}{\sin \theta_0} - 1$$

$$= [1 + f(\theta_0)] \left\{ \frac{\sin \{ \theta_0 [1 + f(\theta_0)] \}}{\sin \theta_0} - 1 \right\}$$

The total strain energy

$$= \frac{1}{2} t E R^2 \int_0^{2\pi} \left\{ [1 + f(\theta_0)] \frac{\sin \{ \theta_0 [1 + f(\theta_0)] \}}{\sin \theta_0} - 1 \right\}^2 d\theta_0$$

Potential energy of the pressure force.

pV where $V = \text{volume under the shell}$

The volume under the shell

3)

$$= \int_0^{\infty} \frac{1}{3} 2\pi R \sin \theta_0 \cdot R \cdot R d\theta_0$$

$$= \frac{2\pi R^3}{3} \int_0^{\infty} [1 + g'(\theta_0)]^3 \sin \{ \theta_0 [1 + f(\theta_0)] \} \{ [1 + f(\theta_0)] + \theta_0 f'(\theta_0) \} d\theta_0$$

The integral to be minimized is

$$\left\{ \frac{1}{R} E \right\} \int_0^{\infty} \left\{ [1 + g'(\theta_0)] \frac{\sin \{ \theta_0 [1 + f(\theta_0)] \}}{\sin \theta_0} - 1 \right\}^2 \sin \theta_0 d\theta_0$$

$$= \frac{2f}{3} \int_0^{\infty} [1 + g'(\theta_0)]^3 \sin \{ \theta_0 [1 + f(\theta_0)] \} \cdot [1 + f(\theta_0) + \theta_0 f'(\theta_0)] d\theta_0$$

To simplify the expression, let us put $\theta_0 f(\theta_0) = h(\theta_0)$

$$\left\{ \begin{aligned} I &= \left\{ \frac{1}{R} E \right\} \int_0^{\infty} \left\{ [1 + g'(\theta_0)] \frac{\sin (\theta_0 + h(\theta_0))}{\sin \theta_0} - 1 \right\}^2 \sin \theta_0 d\theta_0 \\ &= \frac{2f}{3} \int_0^{\infty} [1 + g'(\theta_0)]^3 \sin (\theta_0 + h(\theta_0)) [1 + h'(\theta_0)] d\theta_0 \end{aligned} \right\}$$

The variational condition is

$$\underline{[1 + g'(\theta_0)]^2 [1 + h'(\theta_0)] + [g'(\theta_0)]^2 - 1 = 0}$$

VII Euler-Lagrange

4)

The Euler-Lagrange differential equation is then

$$\begin{aligned} & \frac{t}{R} E \left\{ \cancel{f} \left[[1+g(\theta_0)] \frac{\sin(\theta_0 + h(\theta_0))}{\sin \theta_0} - 1 \right] \cancel{\sin \theta_0} \frac{\sin(\theta_0 + h(\theta_0))}{\cancel{\sin \theta_0}} \right\} \\ & - \frac{\cancel{t}}{3} \left\{ \cancel{f} [1+g(\theta_0)]^2 \sin(\theta_0 + h(\theta_0)) [1+h'(\theta_0)] \right\} \\ & + \lambda \left\{ \cancel{f} [1+g(\theta_0)] [1+h'(\theta_0)]^2 \right\} - \cancel{f} g''(\theta_0) \left\{ \frac{d}{d\theta_0} \left(\lambda \frac{t}{R} \right) \right\} = 0. \\ & \frac{t}{R} E \left\{ \cancel{L} \sin \theta_0 \left[[1+g'(\theta_0)] \frac{\sin(\theta_0 + h(\theta_0))}{\sin \theta_0} - 1 \right] [1+g(\theta_0)] \frac{\cos(\theta_0 + h(\theta_0))}{\sin \theta_0} \right\} \\ & - \frac{\cancel{t}}{3} \left\{ [1+g(\theta_0)]^3 \cos(\theta_0 + h(\theta_0)) [1+h'(\theta_0)] \right. \\ & \left. - 3 [1+g(\theta_0)]^2 g'(\theta_0) \sin(\theta_0 + h(\theta_0)) - [1+g(\theta_0)]^3 \cos(\theta_0 + h(\theta_0)) [1+h'(\theta_0)] \right\} \\ & - \lambda \left\{ \frac{d}{d\theta_0} \left\{ [1+g(\theta_0)]^2 \cancel{f} [1+h'(\theta_0)] \right\} \right\} = 0. \end{aligned}$$

$$\frac{1}{R} E \left\{ [1+g(\theta_0)] \frac{\sin(\theta_0 + k(\theta_0))}{\sin \theta_0} - 1 \right\} \sin(\theta_0 + k(\theta_0)) \quad (1)$$

$$- \frac{1}{3} [1+g(\theta_0)]^2 [1+k'(\theta_0)] \sin(\theta_0 + k(\theta_0))$$

$$+ \lambda \left\{ [1+g(\theta_0)] [1+k'(\theta_0)]^2 - g''(\theta_0) \right\} - g'(\theta_0) \frac{dk}{d\theta_0} = 0 \quad (11)$$

$$\frac{1}{R} E \left\{ [1+g(\theta_0)] \frac{\sin(\theta_0 + k(\theta_0))}{\sin \theta_0} - 1 \right\} [1+g(\theta_0)] \cos(\theta_0 + k(\theta_0))$$

$$- \frac{1}{3} \left\{ [1+g(\theta_0)]^2 \cos(\theta_0 + k(\theta_0)) [1+k'(\theta_0)] \right.$$

$$\left. - 3 [1+g(\theta_0)]^2 g'(\theta_0) \sin(\theta_0 + k(\theta_0)) - [1+g(\theta_0)]^3 [1+k'(\theta_0)] \cos(\theta_0 + k(\theta_0)) \right\}$$

$$- \lambda \left\{ 2 [1+g(\theta_0)] g'(\theta_0) [1+k'(\theta_0)] + [1+g(\theta_0)]^2 k''(\theta_0) \right\} \cos \quad (12)$$

$$- \frac{[1+g(\theta_0)]^2 [1+k'(\theta_0)] \frac{dk}{d\theta_0}}{[1+g(\theta_0)]^2 [1+k'(\theta_0)]^2 + [g'(\theta_0)]^2} = 0$$

$$\frac{[1+g(\theta_0)]^2 [1+k'(\theta_0)]^2 + [g'(\theta_0)]^2}{[1+g(\theta_0)]^2 [1+k'(\theta_0)]^2 + [g'(\theta_0)]^2} - 1 = 0 \quad (13)$$

Now if we assume that both

$$g'(0) \text{ and } h'(0)$$

are small quantities and ~~so the quadratic & higher order~~

~~terms~~ But retain terms of the form $g''(0) g'(0)$

$$\text{Then } \frac{\sin[\theta_0 + h(0)]}{\sin \theta_0} = 1 + \cot \theta_0 \cdot h'(0)$$

[In the following calculation $\theta_0 \approx 0$]

$$[1 + g'(0)] \frac{\sin[\theta + h(\theta)]}{\sin \theta} - 1$$

$$= [1 + g'(0)][1 + h(\theta) \cot \theta] - 1$$

$$\approx g'(0) + h(\theta) \cot \theta$$

$$\sin(\theta + h(\theta)) = \sin \theta + h(\theta) \cos \theta$$

$$\frac{1}{\sin \theta} [1 + g'(0)]^2 [1 + h'(\theta)] [\sin \theta + h(\theta) \cos \theta]$$

$$= [1 + 2g'(0)][1 + h'(\theta)] [\sin \theta + h(\theta) \cos \theta]$$

$$= [1 + 2g'(0) + h'(\theta)] [\sin \theta + h(\theta) \cos \theta]$$

$$= \sin \theta + h(\theta) \cos \theta + 2g'(0) \cdot \sin \theta + h'(\theta) \sin \theta$$

$$[1 + 2g'(0)][1 + 2h'(\theta)] - g''(0)$$

$$= 1 + 2g'(0) + 2h'(\theta) - g''(0)$$

Then the first differential equation becomes

(7)

$$\begin{aligned} & \frac{1}{R} E \left\{ \sin \theta \left[g(\theta) + h'(\theta) \cot \theta \right] \right\} \theta - p \left\{ \sin \theta + h(\theta) \cos \theta \right. \\ & \quad \left. + 2g'(\theta) \sin \theta + h'(\theta) \sin \theta \right\} \\ & + \lambda \left\{ 1 + g'(\theta) + 2h'(\theta) - g''(\theta) \right\} - g'(\theta) \frac{d\lambda}{d\theta} = 0 \end{aligned}$$

$$\cos(\theta + h(\theta)) = \cos \theta - h(\theta) \sin \theta$$

$$[1 + g'(\theta)] \cos(\theta + h(\theta)) = \cos \theta - h(\theta) \sin \theta + g'(\theta) \cos \theta$$

$$\frac{1}{R} E \left\{ \cos \theta \left[g'(\theta) + h'(\theta) \cot \theta \right] \right\} + p \sin \theta \cdot g'(\theta)$$

$$- \frac{p}{R} \left\{ [1 + 2g'(\theta)] \cos \theta - [1 + 2h'(\theta)] \sin \theta + [h'(\theta) \cot \theta + g'(\theta)] \right\}$$

$$- 3 \left[\sin \theta \cdot g'(\theta) \right]$$

$$- \lambda \left\{ 2g'(\theta) + h''(\theta) \right\} - \left\{ 1 + 2g'(\theta) + h'(\theta) \right\} \frac{d\lambda}{d\theta} = 0$$

$$[1 + 2g'(\theta)][1 + 2h'(\theta)] - 1 = 0$$

$$\text{or } \underline{\underline{g'(\theta) + h'(\theta) = 0}}$$

From the last equation, it is seen that $g' \propto h''$ so
 there $h'' \cdot h$, $h''' \cdot h'$, $h'' \cdot h'$ not to be neglected.

$$\frac{t}{R} E [h(\theta) \cos \theta - h'(\theta) \sin \theta] - p \{ \sin \theta + h(\theta) \cos \theta - h'(\theta) \sin \theta \} \\
 - 2h''(\theta) \sin \theta + \lambda \{ 1 + h'(\theta) + h'''(\theta) \} + h''(\theta) \frac{d\lambda}{d\theta} = 0.$$

$$\frac{t}{R} E [h(\theta) \cos \cot \theta - h'(\theta) \cos \theta] - p g''(\theta) \sin \theta \\
 + \lambda [h''(\theta)] - \{ 1 - h'(\theta) \} \frac{d\lambda}{d\theta} = 0.$$

This method of derivation unsatisfactory, because we
 have to use further differentiation to eliminate λ
 but then some of the neglected terms may become
 important.

Timoshenko's differential equation when there is no bending moment will be 9)

$$\frac{dN_x}{d\theta} + (N_x - N_y) \cot \theta + N_y \left(\frac{u}{a} + \frac{1}{a} \frac{dw}{d\theta} \right) = 0$$

$$N_x + N_y + qa + N_x \left(\frac{d^2 w}{a d\theta^2} + \frac{dw}{a d\theta} \right) + N_y \left(\frac{u}{a} + \frac{dw}{a d\theta} \right) \cot \theta = 0$$

$$N_x = \frac{Et}{1-\nu^2} (\epsilon_1 + \nu \epsilon_2) - \frac{qa}{2}$$

$$= \frac{Et}{1-\nu^2} \left[\left(\frac{dw}{a d\theta} - \frac{w}{a} \right) + \nu \left(\frac{u \cot \theta}{a \sin \theta} - \frac{w}{a} \right) \right] - \frac{qa}{2}$$

$$N_y = \frac{Et}{1-\nu^2} (\epsilon_2 + \nu \epsilon_1) - \frac{qa}{2}$$

$$= \frac{Et}{1-\nu^2} \left[\left(\frac{u \cot \theta}{a \sin \theta} - \frac{u}{a} \right) + \nu \left(\frac{dw}{a d\theta} - \frac{w}{a} \right) \right] - \frac{qa}{2}$$

$$\frac{dN_y}{d\theta} = \frac{Et}{1-\nu^2} \left[\frac{1}{a} \frac{d^2 u}{d\theta^2} - \frac{1}{a} \frac{dw}{d\theta} + \nu \left(\frac{1}{a} \cot \theta \frac{dw}{d\theta} + \frac{u}{a} \frac{1}{\sin^2 \theta} - \frac{1}{a} \frac{dw}{d\theta} \right) \right]$$

Hence the differential equation can be written as

$$\frac{Et}{1-\nu^2} \left[\frac{1}{a} \frac{d^2 u}{d\theta^2} - (1+\nu) \frac{1}{a} \frac{dw}{d\theta} + \nu \left(\frac{1}{a} \cot \theta \frac{dw}{d\theta} - \frac{u}{a} \frac{1}{\sin^2 \theta} \right) \right]$$

$$+ \frac{Et}{1-\nu^2} \left[\left(\frac{dw}{a d\theta} - \frac{u \cot \theta}{a \sin \theta} \right) (1-\nu) \right] \cot \theta$$

$$+ \left\{ \frac{Et}{1-\nu^2} \left[\left(\frac{u \cot \theta}{a \sin \theta} - \frac{u}{a} \right) + \nu \left(\frac{dw}{a d\theta} - \frac{w}{a} \right) \right] - \frac{qa}{2} \right\} \left(\frac{u}{a} + \frac{dw}{a d\theta} \right) = 0.$$

Let us put $\rho = \frac{f}{(1-v^2)}$, then

10)

$$\begin{aligned} & \frac{1}{a} \frac{d^2 u}{d\theta^2} - (1+v) \frac{du}{a d\theta} + v \left(\frac{1}{a} \cos\theta \frac{du}{d\theta} - \frac{u}{a} \frac{1}{\sin^2\theta} \right) \\ & + (1-v) \cos\theta \left(\frac{du}{a d\theta} - \frac{u}{a} \cot\theta \right) \\ & + \left(\frac{u}{a} + \frac{du}{a d\theta} \right) \left[\left(-\frac{u}{a} \cot\theta - \frac{uv}{a} \right) + v \left(\frac{du}{a d\theta} - \frac{uv}{a} \right) - \frac{f a}{2} \right] = 0 \end{aligned}$$

~~$\frac{1}{a} \frac{d^2 u}{d\theta^2}$~~

$$\begin{aligned} & \frac{1}{a} \frac{d^2 u}{d\theta^2} - (1+v) \frac{du}{a d\theta} + \cos\theta \frac{du}{a d\theta} - \frac{u}{a} \cot^2\theta - v \frac{u}{a} \\ & - \frac{f a}{2} \left(\frac{u}{a} + \frac{du}{a d\theta} \right) + \left(\frac{u}{a} + \frac{du}{a d\theta} \right) \left[\left(\frac{u}{a} \cot\theta - \frac{uv}{a} \right) + v \left(\frac{du}{a d\theta} - \frac{uv}{a} \right) \right] \end{aligned}$$

$$\begin{aligned} & \frac{1}{a} \frac{d^2 u}{d\theta^2} - \left[1+v + \frac{f a}{2} \right] \frac{du}{a d\theta} + \cos\theta \frac{du}{a d\theta} - \left(\cot^2\theta + v + \frac{f a}{2} \right) \frac{u}{a} \\ & + \left(\frac{u}{a} + \frac{du}{a d\theta} \right) \left[\left(\frac{u}{a} \cot\theta - \frac{uv}{a} \right) + v \left(\frac{du}{a d\theta} - \frac{uv}{a} \right) \right] = 0. \end{aligned}$$

$$\left(\frac{du}{a d\theta} + \frac{u}{a} \cot\theta - \frac{2uv}{a} \right) (1+v) \cancel{v f a}.$$

$$+ \left(\frac{d^2 u}{a d\theta^2} + \frac{du}{a d\theta} \right) \left[\frac{du}{a d\theta} - \frac{uv}{a} + v \left(\frac{u}{a} \cot\theta - \frac{uv}{a} \right) - \frac{f a}{2} \right]$$

$$+ \cos\theta \left(\frac{u}{a} + \frac{du}{a d\theta} \right) \left[\frac{u}{a} \cot\theta - \frac{uv}{a} + v \left(\frac{du}{a d\theta} - \frac{uv}{a} \right) - \frac{f a}{2} \right] = 0.$$

$$\left(\frac{du}{a db} + \frac{u}{a} \cot \theta - \frac{d^2 u}{a} \right) (1+v) - \frac{p_0}{2} \left(\frac{d^2 u}{a db^2} + \frac{du}{a db} + \cot \theta \frac{u}{a} + \cot \theta \frac{du}{a db} \right) \quad (19)$$

$$\begin{aligned} & \cancel{\left(1+v - \frac{p_0}{2} \right) \frac{du}{a db} + \left(1+v - \frac{p_0}{2} \right) \frac{u}{a} \cot \theta - \frac{p_0}{2} \frac{d^2 u}{a db^2} - \frac{p_0}{2} \cot \theta \frac{du}{a db}} \\ & = (1+v) \frac{2u}{a} \end{aligned}$$

$$\begin{aligned} & \left(1+v - \frac{p_0}{2} \right) \left(\frac{du}{a db} + \frac{u}{a} \cot \theta \right) - \frac{p_0}{2} \frac{d^2 u}{a db^2} - \frac{p_0}{2} \cot \theta \frac{du}{a db} - (1+v) \frac{2u}{a} \\ & + \left(\frac{d^2 u}{a db^2} + \frac{du}{a db} \right) \left[\frac{du}{a db} - \frac{u}{a} + v \left(\frac{u}{a} \cot \theta - \frac{u}{a} \right) \right] \\ & + \cot \theta \left(\frac{u}{a} + \frac{du}{a db} \right) \left[\frac{u}{a} \cot \theta - \frac{u}{a} + v \left(\frac{du}{a db} - \frac{u}{a} \right) \right] = 0 \end{aligned}$$

Neglect the change in curvature in y -direction, we have

$$\cancel{\frac{1}{2} \frac{d^2 u}{db^2} - \left[1+v + \frac{p_0}{2} \right] \frac{du}{a db} + \cot \theta \frac{du}{a db} - \left(\cot^2 \theta + v + \frac{p_0}{2} \right) \frac{u}{a}}$$

$$\frac{1}{2} \frac{d^2 u}{db^2} - \left[1+v + \frac{p_0}{2} \right] \frac{du}{a db} + \cot \theta \frac{du}{a db} - \left(\cot^2 \theta + v + \frac{p_0}{2} \right) \frac{u}{a} = 0$$

$$\begin{aligned} & \left(1+v - \frac{p_0}{2} \right) \left(\frac{du}{a db} + \frac{u}{a} \cot \theta \right) - \frac{p_0}{2} \left(\frac{d^2 u}{a db^2} + \cot \theta \frac{du}{a db} \right) - (1+v) \frac{2u}{a} \\ & + \left(\frac{d^2 u}{a db^2} + \frac{du}{a db} \right) \left[\frac{du}{a db} - \frac{u}{a} + v \left(\frac{u}{a} \cot \theta - \frac{u}{a} \right) \right] = 0. \end{aligned}$$

Putting $u = \frac{d\psi}{d\theta}$, we have from the first equation (12)
by integrating

$$\frac{d^2\psi}{d\theta^2} + \cot\theta \frac{d\psi}{d\theta} + 2\psi - (1+\nu)(\psi+w) - \frac{p_2}{2}(\psi+w) = 0$$

Similarly, by putting $u = \frac{d\psi}{d\theta}$ into the second equation, we have

$$(1+\nu - \frac{p_2}{2}) \left(\frac{d^2\psi}{d\theta^2} + \cot\theta \frac{d\psi}{d\theta} \right) - \frac{p_2}{2} \left(\frac{d^2w}{d\theta^2} + \cot\theta \frac{dw}{d\theta} \right) - (1+\nu) 2w + \left(\frac{d^2w}{d\theta^2} + \frac{d^2\psi}{d\theta^2} \right) \left(\frac{d\psi}{d\theta^2} - \frac{w}{\theta} (1+\nu) + \nu \cot\theta \frac{d\psi}{d\theta} \right) = 0$$

Let $\frac{u}{a} = v(u)$, $\frac{w}{a} = v(w)$, $\frac{p_2}{2} = \phi$.

$$\frac{d^2\psi}{d\theta^2} + \cot\theta \frac{d\psi}{d\theta} + 2\psi - (1+\nu)(\psi+w) - \phi(\psi+w) = 0$$

$$(1+\nu - \phi) \left(\frac{d^2\psi}{d\theta^2} + \cot\theta \frac{d\psi}{d\theta} \right) - \phi \left(\frac{d^2w}{d\theta^2} + \cot\theta \frac{dw}{d\theta} \right) - 2(1+\nu)w + \left(\frac{d^2w}{d\theta^2} + \frac{d^2\psi}{d\theta^2} \right) \left(\frac{d\psi}{d\theta^2} + \nu \cot\theta \frac{d\psi}{d\theta} - (1+\nu)w \right) = 0.$$

$$\begin{aligned} \approx & (1+\nu) \left(\frac{d^2\psi}{d\theta^2} + \cot\theta \frac{d\psi}{d\theta} \right) - \phi \left(\frac{d^2(\psi+w)}{d\theta^2} + \cot\theta \frac{d(\psi+w)}{d\theta} \right) - 2(1+\nu)w \\ & + \left[\frac{d^2(\psi+w)}{d\theta^2} \right] \left[\frac{d\psi}{d\theta^2} + \nu \cot\theta \frac{d\psi}{d\theta} - (1+\nu)w \right] = 0 \end{aligned}$$

$$\frac{d^2\psi}{d\theta^2} + \cot\theta \frac{d\psi}{d\theta} + 2\psi - [(1+v)+\phi]\gamma = 0 \quad (3)$$

$$(1+v)[-1+v+\phi]\gamma - \phi \left[\frac{d^2\gamma}{d\theta^2} + \cot\theta \frac{d\gamma}{d\theta} \right] - 2(1+v)\gamma \\ + \frac{d^2\gamma}{d\theta^2} \left[\frac{d^2\phi}{d\theta^2} + v \cot\theta \frac{d\phi}{d\theta} + (1+v)\psi - (1+v)\gamma \right] = 0$$

$$\frac{d^2\psi}{d\theta^2} + \cot\theta \frac{d\psi}{d\theta} + 2\psi - [1+v+\phi]\gamma = 0$$

$$\frac{d^2\gamma}{d\theta^2} \left[- (1-v) \cot\theta \frac{d\psi}{d\theta} - (1-v)\psi + \phi\gamma \right] + (1+v)[\phi - (1-v)]\gamma \\ - \phi \left[\frac{d^2\gamma}{d\theta^2} + \cot\theta \frac{d\gamma}{d\theta} \right] = 0.$$

$$\frac{d^2\psi}{d\theta^2} + \cot\theta \frac{d\psi}{d\theta} + 2\psi - [1+v+\phi]\gamma = 0$$

$$(1+v)[\phi - (1-v)]\gamma - \frac{d^2\gamma}{d\theta^2} \left[(1-v) \cot\theta \frac{d\psi}{d\theta} + (1-v)\psi - \phi\gamma \right] - \phi \left[\frac{d^2\gamma}{d\theta^2} + \cot\theta \frac{d\gamma}{d\theta} \right] = 0$$

neglecting the curvature terms, we have

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$$H(\psi) - (1+\nu)(\psi + w) - \phi(\psi + w) = 0.$$

$$(1+\nu-\phi) \left(\frac{d^2\psi}{dt^2} + \cos\theta \frac{d\psi}{dt} \right) - \phi \left(\frac{d^2w}{dt^2} + \cos\theta \frac{dw}{dt} \right) - (1+\nu) \frac{dw}{dt} = 0$$

$$(1+\nu-\phi) [H(\psi) - 2\psi] - \phi [H(w) - 2w] - (1+\nu) \frac{dw}{dt} = 0$$

Put in $\psi = \sum_{n=0}^{\infty} A_n P_n$

$$w = \sum_{n=0}^{\infty} B_n P_n$$

$$\sum_{n=0}^{\infty} [-A_n \lambda_n - (1+\nu+\phi)(A_n + B_n)] P_n = 0$$

$$\sum_{n=0}^{\infty} \left[(1+\nu-\phi) [-\lambda_n A_n - 2A_n] - \phi [-\lambda_n B_n - 2B_n] - 2(1+\nu) B_n \right] P_n = 0$$

$$\sum_{n=0}^{\infty} \left[(1+\nu+\phi+\lambda_n) A_n + (1+\nu+\phi) B_n \right] P_n = 0$$

$$\sum_{n=0}^{\infty} \left[(1+\nu-\phi)(2+\lambda_n) A_n - \{ \phi(2+\lambda_n) - 2(1+\nu) \} B_n \right] P_n = 0.$$

The set of homogeneous equation for A_n and B_n is

$$(1+r+\phi+\lambda_n)A_n + (1+r+\phi)B_n = 0$$

$$(1+r-\phi)(2+\lambda_n)A_n + \{2(1+r)-\phi(2+\lambda_n)\}B_n = 0$$

The determinant must be zero, so

$$(1+r+\phi+\lambda_n)\{2(1+r)-\phi(2+\lambda_n)\}$$

$$- (1+r+\phi)(1+r-\phi)(2+\lambda_n) = 0.$$

$$2(1+r+\phi+\lambda_n)(1+r) - (1+r+\lambda_n)(2+\lambda_n)\phi - \phi^2(2+\lambda_n)$$

$$- (2+\lambda_n)(1+r+\phi)(1+r) + (2+\lambda_n)(1+r)\phi + \phi^2(2+\lambda_n) = 0$$

$$2(1+r)^2 + 2(1+r)(\phi+\lambda_n) - (2+\lambda_n)\lambda_n\phi$$

$$- 2(1+r)^2 - \lambda_n(1+r+\phi)(1+r) - 2\phi(1+r) = 0$$

$$\frac{2(1+r)\lambda_n}{2(1+r)\phi + 2(1+r)\lambda_n} = \frac{(2+\lambda_n)\lambda_n\phi - \lambda_n(1+r)\phi - \lambda_n(1+r)^2}{2(1+r)\phi + 2(1+r)\lambda_n} = 0$$

$$2(1+r)^2 + 2(1+r)(\phi+\lambda_n) - \lambda_n(2+\lambda_n)\phi$$

$$- 2(1+r)^2 - 2(1+r)\phi - \lambda_n(1+r)^2 - \lambda_n\phi(1+r) = 0.$$

$$\lambda_n(1+r)[2 - (1+r)] - \lambda_n\phi[2 + \lambda_n + 1+r] = 0$$

$$(1+r)(1-r) - \phi[2 + \lambda_n + 1+r] = 0.$$

$$\phi = \frac{1-r^2}{3+r+\lambda_n} \quad \text{When } \phi = 0$$

$$\frac{pa}{2} = \frac{qa(1-\nu^2)}{2Et} = \frac{1-\nu^2}{3+\nu+\lambda_n}$$

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$$\frac{q}{2E} \left(\frac{a}{t} \right) = \frac{1}{3+\nu+\lambda_n}$$

$$q_{cr} = \frac{2E}{3+\nu+\lambda_n} \left(\frac{t}{a} \right)$$

$$\tau_{cr} = \frac{2E}{3+\nu+\lambda_n} \left(\frac{t}{a} \right) \left(\frac{a}{t} \right) \frac{1}{2} = \left(\frac{E}{3+\nu+\lambda_n} \right)$$



$$R = \frac{R_0}{1 + \frac{w}{a}}$$

$$R = a + \frac{w}{a} = 2 - \frac{w}{a}$$

$$\theta_1 = \theta_0 + \frac{w}{a}$$

The original length of the element $(ds)_0 = a (d\theta_0)$
 The new length of the element

$$= \sqrt{R^2 (d\theta)^2 + (dn)^2} = a \sqrt{\left(1 + \frac{w}{a}\right)^2 \left(1 + \frac{1}{a} \frac{dw}{d\theta}\right)^2 + \left(\frac{dw}{d\theta}\right)^2} d\theta$$

~~neglecting quadratic terms of deflections~~

$$= a d\theta \left\{ 1 + \frac{w}{a} \left(2 + \frac{w}{a}\right) + \frac{1}{a} \frac{dw}{d\theta} \left(2 + \frac{1}{a} \frac{dw}{d\theta}\right) + \frac{1}{a^2} \left(\frac{dw}{d\theta}\right)^2 \right\}^{\frac{1}{2}}$$

The distance of the element from the axis is

$$\frac{a \sin \theta_0}{2} \quad \text{before deflection}$$

The distance is $\frac{a \sin \theta}{2}$ after deflection

The change in length of the arc ds is

$$2\pi a \left[\left(1 + \frac{w}{a}\right) \sin \left(\theta + \frac{w}{a}\right) - \sin \theta \right]$$

latitude)

The change per unit length (circumferential)

18)

$$= \frac{(1 + \frac{w}{a}) \sin(\theta + \frac{u}{a})}{\sin \theta} - 1$$

The change per unit length (meridian)

$$\left\{ 1 + \frac{1}{a} \frac{du}{d\theta} \left(2 + \frac{1}{a} \frac{du}{d\theta} \right) + \frac{w}{a} \left(2 + \frac{w}{a} \right) + \frac{1}{a} \frac{dw}{d\theta} \left(\frac{1}{a} \frac{dw}{d\theta} \right) \right\}^{\frac{1}{2}} - 1$$

$$\approx \frac{1}{a} \frac{du}{d\theta} \left(1 + \frac{1}{2a} \frac{du}{d\theta} \right) + \frac{w}{a} \left(1 + \frac{w}{2a} \right) + \frac{1}{a} \frac{dw}{d\theta} \left(\frac{1}{2a} \frac{dw}{d\theta} \right)$$

$$- \frac{1}{2} \frac{1}{a^2} \left(\frac{du}{d\theta} \right)^2 - \frac{1}{2} \frac{1}{a^2} w^2 - \frac{1}{2a^2} \left(\frac{dw}{d\theta} \right)^2$$

$$\approx \frac{1}{a} \frac{du}{d\theta} + \frac{w}{a} + \frac{1}{2a^2} \frac{du^2}{d\theta^2} + \frac{1}{2a^2} \frac{dw^2}{d\theta^2} \quad \left[\text{up to second order terms} \right]$$

$$\frac{(1 + \frac{w}{a}) \sin(\theta + \frac{u}{a})}{\sin \theta} - 1$$

$$= (1 + \frac{w}{a}) \left[\cos(\frac{u}{a}) + \cot \theta \sin(\frac{u}{a}) \right] - 1$$

$$\approx (1 + \frac{w}{a}) \left[1 - \frac{1}{2} \left(\frac{u}{a} \right)^2 + \cot \theta \cdot \left(\frac{u}{a} \right) \right] - 1$$

$$\approx -\frac{1}{2} \left(\frac{u}{a} \right)^2 + \cot \theta \left(\frac{u}{a} \right) + \cot \theta \left(\frac{u}{a} \right) \left(\frac{w}{a} \right) + \frac{w}{a}$$

The stress in latitude direction

$$\frac{E}{1-\nu^2} \left[\frac{w}{a} + \frac{u}{a} \cot \theta - \frac{1}{2} \left(\frac{u}{a} \right)^2 + \frac{2uw}{a^2} \cot \theta + \nu \right] \left[\frac{1}{a} \frac{du}{d\theta} + \frac{uw}{a} + \frac{1}{a^2} \left(\frac{dw}{d\theta} \right)^2 \right] \\ - \frac{pa}{2t}$$

The stress in meridian direction

$$\frac{E}{1-\nu^2} \left[\frac{1}{a} \frac{du}{d\theta} + \frac{w}{a} + \frac{1}{a^2} \left(\frac{dw}{d\theta} \right)^2 + \nu \right] \left[\frac{uw}{a} + \frac{u}{a} \cot \theta - \frac{1}{2} \left(\frac{u}{a} \right)^2 + \frac{2uw}{a^2} \cot \theta \right] \\ - \frac{pa}{2t}$$

The strain energy retaining only terms up to second order

$$2\pi a^2 t \frac{E}{1-\nu^2} \left\{ \frac{1}{2} \int \left[\frac{1}{a} \frac{du}{d\theta} + \frac{w}{a} + \nu \right] \left[\frac{uw}{a} + \frac{u}{a} \cot \theta \right] \left[\frac{1}{a} \frac{du}{d\theta} + \frac{uw}{a} \right] \sin \theta d\theta \right. \\ \left. - \frac{1}{2} \int \left[\frac{uw}{a} + \frac{u}{a} \cot \theta + \nu \right] \left[\frac{1}{a} \frac{du}{d\theta} + \frac{uw}{a} \right] \left[\frac{uw}{a} + \frac{u}{a} \cot \theta \right] \sin \theta d\theta \right\} \\ - \frac{pa}{2t} 2\pi a^2 t \int \left[\frac{uw}{a} + \frac{u}{a} \cot \theta - \frac{1}{2} \left(\frac{u}{a} \right)^2 + \frac{2uw}{a^2} \cot \theta \right] \sin \theta d\theta \\ - \frac{pa}{2t} 2\pi a^2 t \int \left[\frac{1}{a} \frac{du}{d\theta} + \frac{w}{a} + \frac{1}{a^2} \left(\frac{dw}{d\theta} \right)^2 \right] \sin \theta d\theta$$

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The volume under the shell

$$= \frac{2\pi a^3}{3} \int \left[1 + \frac{u}{a}\right]^3 \left[\sin \theta \left[1 - \frac{1}{2} \left(\frac{u}{a}\right)^2\right] + \cos \theta \left(\frac{u}{a}\right) \right] \left[1 + \frac{1}{a} \frac{du}{d\theta}\right] d\theta$$

$$= \frac{2\pi a^3}{3} \int \left\{ 1 + 3 \frac{u}{a} + 3 \left(\frac{u}{a}\right)^2 \right\} \left\{ \left[1 - \frac{1}{2} \left(\frac{u}{a}\right)^2\right] \sin \theta + \left(\frac{u}{a}\right) \cos \theta \right\} \left[1 + \frac{1}{a} \frac{du}{d\theta}\right] d\theta$$

$$\sim \frac{2\pi a^3}{3} \int \left[\sin \theta \left(1 + 3 \frac{u}{a} + 3 \frac{u^2}{a^2}\right) - \frac{1}{2} \left(\frac{u}{a}\right)^2 \sin \theta + \left(\frac{u}{a}\right) \cos \theta \right. \\ \left. + 3 \cos \theta \frac{u}{a^2} \right] \left[1 + \frac{1}{a} \frac{du}{d\theta}\right] d\theta.$$

$$\sim \frac{2\pi a^3}{3} \int \left[\sin \theta \left(3 \frac{u}{a} + 3 \frac{u^2}{a^2}\right) - \frac{1}{2} \left(\frac{u}{a}\right)^2 \sin \theta + \left(\frac{u}{a}\right) \cos \theta + 3 \cos \theta \frac{u}{a^2} \right. \\ \left. + \frac{1}{a} \frac{du}{d\theta} \left(\sin \theta + 3 \sin \theta \frac{u}{a} + \left(\frac{u}{a}\right) \cos \theta \right) \right] d\theta$$

The integral to be minimized is

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$$\left\{ \frac{t}{a} \frac{E}{1-y^2} \right\} \int_0^{\pi} \left\{ \frac{1}{2} \left[\left(\frac{1}{a} \frac{du}{dt} + \frac{u}{a} \right)^2 + \left(\frac{u}{a} + \frac{u}{a} \cos \theta \right)^2 + 2 \sqrt{\left(\frac{1}{a} \frac{du}{dt} + \frac{u}{a} \right) \left(\frac{u}{a} + \frac{u}{a} \cos \theta \right)} \right] \right\} \sin \theta d\theta$$

$$= \frac{t}{2} \int_0^{\pi} \left\{ \frac{u^2}{a^2} + 2 \cos \theta \frac{u}{a} + \frac{1}{a^2} \left(\frac{du}{dt} \right)^2 + \frac{u^2}{a^2} + 2 \cos \theta \frac{u}{a} + \frac{1}{a^2} \left(\frac{du}{dt} \right)^2 + \frac{2u}{a^2} \cos \theta \left(\frac{du}{dt} + \frac{u}{a} \right) \right\} \sin \theta d\theta$$

$$= \frac{t}{3} \int_0^{\pi} \left\{ \frac{3u^2}{a^2} + 2 \left(\frac{u}{a} \right)^2 - \frac{1}{2} \left(\frac{u}{a} \right)^2 + \cos \theta \frac{u}{a} + \frac{1}{2} \cos \theta \frac{u^2}{a^2} + \frac{1}{a} \frac{du}{dt} \left(1 + 3 \frac{u}{a} + \cos \theta \frac{u}{a} \right) \right\} \sin \theta d\theta$$

The integral to be minimized is

$$\left(\frac{t}{a} \frac{E}{1-y^2} \right) \int_0^{\pi} \left\{ \frac{1}{2} \left[\left(\frac{1}{a} \frac{du}{dt} + \frac{u}{a} \right)^2 + \left(\frac{u}{a} + \frac{u}{a} \cos \theta \right)^2 + 2 \sqrt{\left(\frac{1}{a} \frac{du}{dt} + \frac{u}{a} \right) \left(\frac{u}{a} + \frac{u}{a} \cos \theta \right)} \right] \right\} \sin \theta d\theta$$

$$\left[\frac{t}{2} \int_0^{\pi} \left\{ \frac{u^2}{a^2} + \frac{5}{6} \cos \theta \frac{u}{a} + \frac{1}{6} \cos \theta \frac{1}{a} \frac{du}{dt} - \frac{1}{12} \left(\frac{u}{a} \right)^2 + \frac{1}{2a^2} \left(\frac{du}{dt} \right)^2 + \frac{3}{2} \frac{u^2}{a^2} \cos \theta + \cos \theta \frac{1}{a^2} u \frac{du}{dt} \right\} \sin \theta d\theta \right]$$

$$= \frac{t}{2} \int_0^{\pi} \left\{ \frac{u^2}{a^2} + \frac{u^2}{a^2} - \frac{1}{6} \left(\frac{u}{a} \right)^2 + \frac{1}{5} \left(\frac{u}{a} \right) \cos \theta + \frac{u^2}{a^2} \cos \theta + \frac{1}{a} \frac{du}{dt} \left(\frac{1}{3} + \frac{u}{a} + \cos \theta \frac{u}{a} \right) \right\} \sin \theta d\theta$$

Bending without extension in meridian

We have the differential equations

$$\frac{dN_x'}{d\theta} + (N_x' - N_y') \cot \theta - Q_x - \frac{qa}{2} \left(\frac{u}{a} + \frac{dw}{a d\theta} \right) = 0$$

$$\begin{aligned} \frac{dQ_x}{d\theta} + Q_x \cot \theta + N_x' + N_y' + qa \left(\frac{du}{a d\theta} + \frac{u}{a} \cot \theta - \frac{dw}{a} \right) \\ - \frac{qa}{2} \left(\frac{du}{a d\theta} + \frac{d^2 w}{a d\theta^2} \right) - \frac{qa}{2} \cot \theta \left(\frac{u}{a} + \frac{dw}{a d\theta} \right) = 0 \end{aligned}$$

$$\frac{dM_x}{d\theta} + (M_x - M_y) \cot \theta - Q_x a = 0.$$

Putting $w = \frac{du}{d\theta}$

$$N_x' = \frac{Eh}{1-\nu^2} \left\{ \nu \left(\frac{u \cot \theta}{a} - \frac{du}{a d\theta} \right) \right\}$$

$$N_y' = \frac{Eh}{1-\nu^2} \left\{ \frac{u \cot \theta}{a} - \frac{du}{a d\theta} \right\}$$

$$M_x' = -\frac{D}{a^2} \left[\frac{du}{d\theta} + \frac{d^2 u}{d\theta^2} + \nu \left(a + \frac{d^2 u}{d\theta^2} \right) \cot \theta \right]$$

$$M_y = -\frac{D}{a^2} \left[\left(1 + \frac{d^2 u}{d\theta^2} \right) \cot \theta + \nu \left(\frac{du}{d\theta} + \frac{d^2 u}{d\theta^2} \right) \right]$$

23)

$$\frac{dN_x'}{dt} + (N_x' - N_y') \cot \theta - Q_x - \frac{qa}{2} \left(\frac{u}{a} + \frac{1}{a} \frac{d^2 u}{dt^2} \right) = 0$$

$$\frac{dQ_x}{dt} + Q_x \cot \theta + N_x' + N_y' + \frac{qa}{2} \left[-\frac{du}{adt} + \frac{u}{a} \cot \theta - \frac{1}{2} \frac{du}{adt} - \frac{1}{2} \frac{d^3 u}{adt^3} - \frac{\cot \theta}{2} \left(\frac{u}{a} + \frac{d^2 u}{adt^2} \right) \right] = 0.$$

$$\sim \boxed{\frac{dQ_x}{dt} + Q_x \cot \theta + N_x' + N_y' + \frac{qa}{2} \left[\cot \theta \left(\frac{u}{a} - \frac{d^2 u}{adt^2} \right) - 3 \frac{du}{adt} - \frac{d^3 u}{adt^3} \right] = 0.}$$

$$Q_x = \frac{1}{a} \left\{ \frac{dM_x}{dt} + (M_x - M_y) \cot \theta \right\}$$

$$1 + \cot^2 \theta = \csc^2 \theta$$

$$\frac{1}{a} \left\{ \frac{d^2 M_x}{dt^2} + \left(\frac{dM_x}{dt} - \frac{dM_y}{dt} \right) \cot \theta \right\}$$

$$= -\frac{D}{a^3} \left\{ \frac{d^2 u}{dt^2} + \frac{d^4 u}{dt^4} + v \left(\frac{du}{dt} + \frac{d^3 u}{dt^3} \right) \cot \theta - v \left(u + \frac{d^2 u}{dt^2} \right) \csc^2 \theta \right\}$$

$$= \left\{ (1-v) \cot \theta \left(u - \frac{du}{dt} + \frac{d^2 u}{dt^2} - \frac{d^3 u}{dt^3} \right) \right\}$$

$$Q_x = -\frac{D}{a^3} \left\{ \frac{d^2 u}{dt^2} + \frac{d^4 u}{dt^4} + v \left(\frac{du}{dt} + \frac{d^3 u}{dt^3} \right) \cot \theta - v \left(u + \frac{d^2 u}{dt^2} \right) \csc^2 \theta \right\}$$

$$= \csc^2 \theta \left(u - \frac{du}{dt} + \frac{d^2 u}{dt^2} - \frac{d^3 u}{dt^3} \right) + v \left(u - \frac{du}{dt} + \frac{d^2 u}{dt^2} - \frac{d^3 u}{dt^3} \right) \cot \theta$$

$$= -\frac{D}{a^3} \left\{ \frac{d^2 u}{dt^2} + \frac{d^4 u}{dt^4} - v \left(u + \frac{d^2 u}{dt^2} \right) \csc^2 \theta - (1-v) \cot \theta \left(u + \frac{d^2 u}{dt^2} \right) + \cot \theta \left(\frac{du}{dt} + \frac{d^3 u}{dt^3} \right) \right\}$$

$$\begin{aligned}
& - \frac{D}{a^3} \left[\frac{d^3 u}{db^3} + \frac{d^5 u}{db^5} - v \left(\frac{du}{db} + \frac{d^3 u}{db^3} \right) \cos^2 \theta + v \left(u + \frac{d^2 u}{db^2} \right) \cos \theta \cot \theta \right. \\
& \quad + (1-v) \left(u + \frac{d^2 u}{db^2} \right) - (1-v) \cot \theta \left(\frac{du}{db} + \frac{d^3 u}{db^3} \right) \\
& \quad - \left(\frac{du}{db} + \frac{d^3 u}{db^3} \right) + 2 \cot \theta \left(\frac{d^2 u}{db^2} + \frac{d^4 u}{db^4} \right) \\
& \quad + \cot \theta \left(\frac{d^2 u}{db^2} + \frac{d^4 u}{db^4} \right) - v \left(u + \frac{d^2 u}{db^2} \right) \cos^2 \theta - v \\
& \quad \left. - \frac{(1-v) \cot^2 \theta \left(u + \frac{d^2 u}{db^2} \right) + \cot^2 \theta \left(\frac{du}{db} + \frac{d^3 u}{db^3} \right)}{2} \right] \\
& + \frac{Eh}{1-v^2} (1+v) \left[\frac{u \cot \theta}{a} - \frac{du}{a db} \right] + \frac{qa}{2} \left[\cot \theta \left(\frac{u}{a} - \frac{d^2 u}{db^2} \right) \right. \\
& \quad \left. - 3 \frac{du}{a db} - \frac{d^3 u}{a db^3} \right] = 0
\end{aligned}$$

$$\begin{aligned}
& - \frac{D}{a^3} \left[\frac{d^3 u}{db^3} + \frac{d^5 u}{db^5} - (1+v) \left(\frac{du}{db} + \frac{d^3 u}{db^3} \right) - \left\{ (1-v) \cot \theta + v \cot^2 \theta \right\} \left(\frac{du}{db} + \frac{d^3 u}{db^3} \right) \right. \\
& \quad + 2 \cot \theta \left(\frac{d^2 u}{db^2} + \frac{d^4 u}{db^4} \right) + \left\{ (1-v) + v \cot \theta + \cot^2 \theta \right\} \left(u + \frac{d^2 u}{db^2} \right) \left. \right] \\
& + \frac{Eh}{1-v^2} (1+v) \left[\frac{u \cot \theta}{a} - \frac{du}{a db} \right] + \frac{qa}{2} \left[\cot \theta \left(\frac{u}{a} - \frac{d^2 u}{db^2} \right) \right. \\
& \quad \left. - 3 \frac{du}{a db} - \frac{d^3 u}{a db^3} \right] = 0.
\end{aligned}$$

$$\begin{aligned}
& -\alpha \left[\frac{d^3 u}{dt^3} + \frac{d^3 u}{dt^3} - (1-\nu) \left(\frac{du}{dt} + \frac{d^3 u}{dt^3} \right) - \{ (1-\nu) \cos \theta + \nu \cos^3 \theta \} \left(\frac{du}{dt} + \frac{d^3 u}{dt^3} \right) \right. \\
& \quad \left. + 2 \cos \theta \left(\frac{d^2 u}{dt^2} + \frac{d^2 u}{dt^2} \right) + \{ (1-\nu) + \nu (\cos^2 \theta + \cos^4 \theta) \} \left(u + \frac{d^4 u}{dt^4} \right) \right] \\
& + (1+\nu) \left[\cos \theta \frac{du}{dt} - \frac{d^3 u}{dt^3} \right] + \phi \left[\left(u - \frac{d^2 u}{dt^2} \right) \cos \theta - 3 \frac{du}{dt} - \frac{d^3 u}{dt^3} \right] = 0.
\end{aligned}$$

Let $u = \frac{d^2 \psi}{dt^2}$

$$\begin{aligned}
& -\alpha \left[\frac{d^6 \psi}{dt^6} + 2 \cos \theta \frac{d^5 \psi}{dt^5} - \{ 4 \cos^2 \theta + (1-\nu) \cos \theta \} \frac{d^4 \psi}{dt^4} \right. \\
& \quad \left. + \{ (1-\nu) + (2+\nu) \cos \theta + \nu \cos^3 \theta \} \frac{d^3 \psi}{dt^3} \right. \\
& \quad \left. - \{ (1+\nu) + (1-\nu) \cos \theta + \nu \cos^2 \theta \} \frac{d^2 \psi}{dt^2} \right. \\
& \quad \left. + \{ (1-\nu) + \nu (\cos \theta + \cos^3 \theta) \} \frac{d \psi}{dt} \right] \\
& + (1+\nu) \left[\cos \theta \frac{d \psi}{dt} - \frac{d^3 \psi}{dt^3} \right] + \phi \left[\left(\frac{d \psi}{dt} - \frac{d^3 \psi}{dt^3} \right) \cos \theta - 3 \frac{d \psi}{dt} - \frac{d^4 \psi}{dt^4} \right] = 0.
\end{aligned}$$

Pure Bending Change in curvature $\frac{1}{a}$

Strain energy $\sim (\text{Change in curvature})^2 \times \text{bending stiffness} \times \text{area} \sim \left(\frac{1}{a}\right)^2 Et^3 \cdot a^2 \sim Et^3$

Potential energy $\sim \rho a^3$

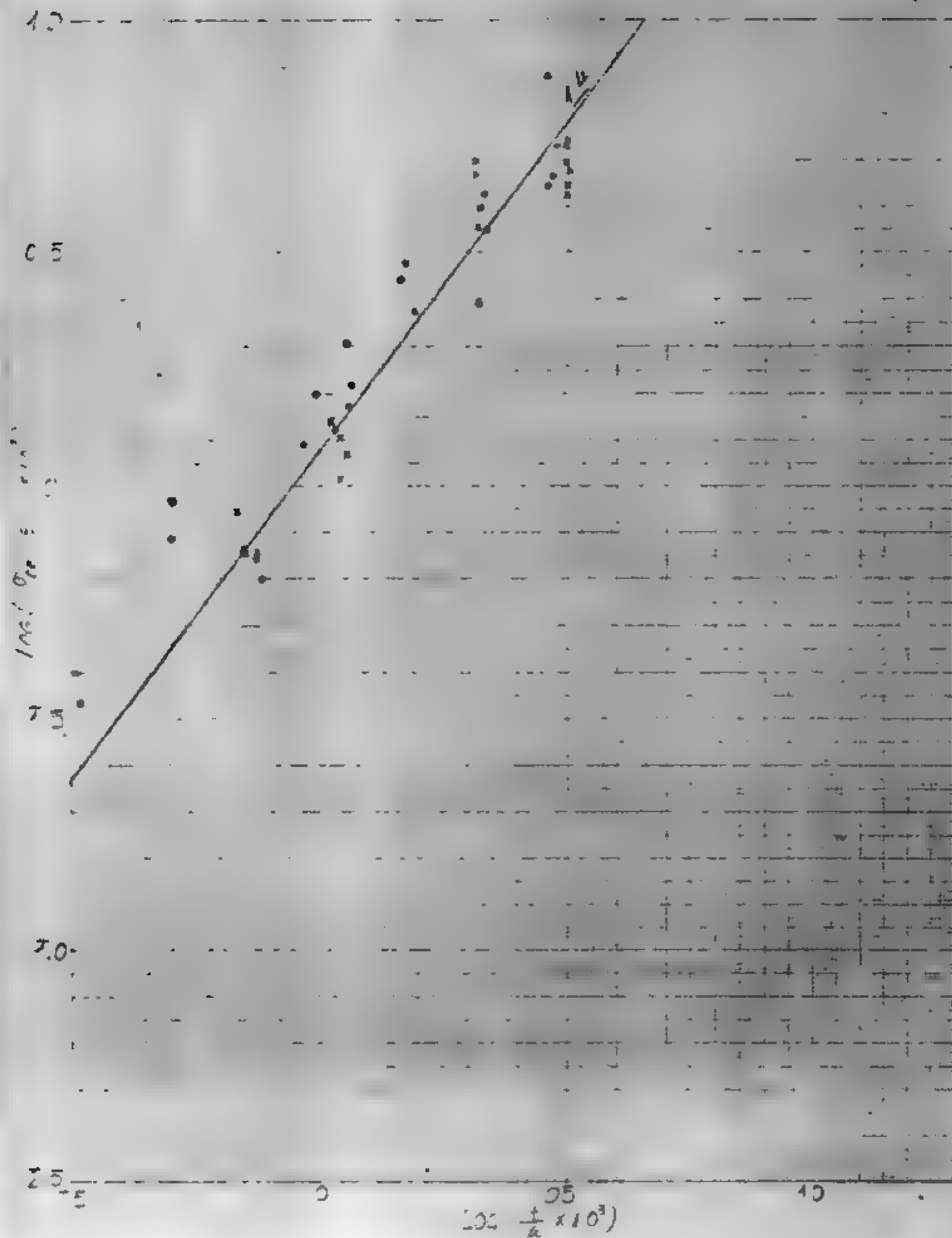
$$f_{cr} \sim E \left(\frac{t}{a}\right)^3$$

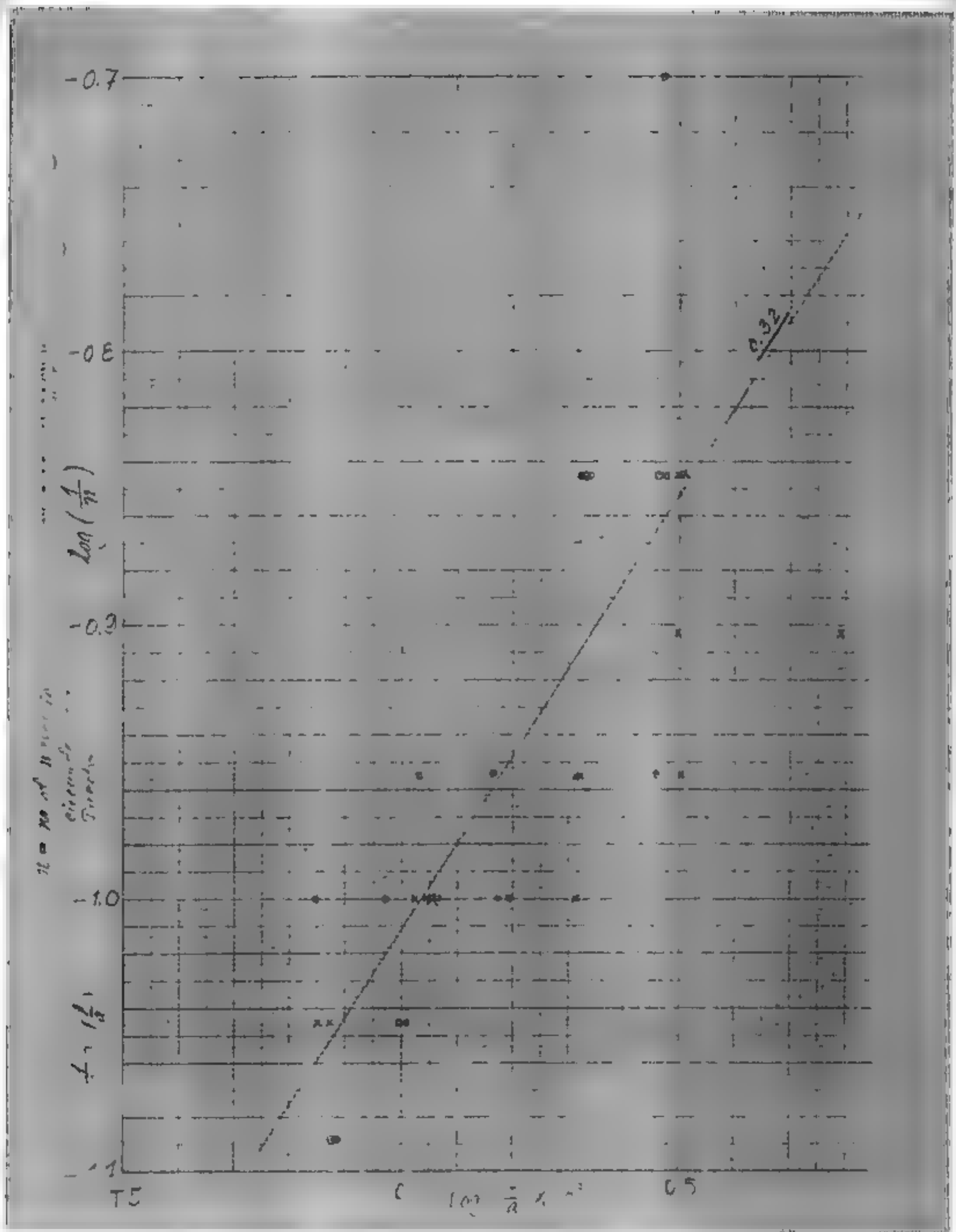
$$\underline{\underline{\sigma_{cr} \sim E \left(\frac{t}{a}\right)^2}}$$

B nara

	$\frac{t}{a}$	A	σ_w/E	$\frac{t}{a}$	A	σ_w/E	
1	1.015×10^{-3}	0.00512	1.484×10^{-4}	2.060×10^{-3}	0.01037	3.96×10^{-4}	27
2	0.981×10^{-3}	0.00494	1.586×10^{-4}	3.160×10^{-3}	0.00701	4.40×10^{-4}	28
3	1.556×10^{-3}	0.00334	2.355×10^{-4}	3.95×10^{-3}	0.00694	4.21×10^{-4}	29
4	1.443×10^{-3}	0.00327	2.99×10^{-4}	6.19×10^{-3}	0.00345	12.12×10^{-4}	30
5	3.012×10^{-3}	0.00168	5.36×10^{-4}	1.05×10^{-3}	0.00530	1.38×10^{-4}	31
6	2.815×10^{-3}	0.00161	7.58×10^{-4}	3.13×10^{-3}	0.00175	4.94×10^{-4}	32
7	0.765×10^{-3}	0.00386	0.63×10^{-4}	3.96×10^{-3}	0.00175	5.54×10^{-4}	33
8	0.744×10^{-3}	0.00375	0.70×10^{-4}	0.744×10^{-3}	0.00376	0.72×10^{-4}	34
9	1.162×10^{-3}	0.00257	1.65×10^{-4}	0.710×10^{-3}	0.00357	0.72×10^{-4}	35
10	1.137×10^{-3}	0.00250	2.03×10^{-4}	1.136×10^{-3}	0.00251	1.17×10^{-4}	36
11	2.175×10^{-3}	0.00121	3.54×10^{-4}	1.10×10^{-3}	0.00244	1.03×10^{-4}	37
12	2.155×10^{-3}	0.00120	4.23×10^{-4}	2.09×10^{-3}	0.01060	3.60×10^{-4}	38
13	0.932×10^{-3}	0.00470	1.23×10^{-4}	2.06×10^{-3}	0.01044	5.02×10^{-4}	39
14	1.460×10^{-3}	0.00322	2.75×10^{-4}	2.06×10^{-3}	0.01044	4.79×10^{-4}	40
15	2.906×10^{-3}	0.00162	4.46×10^{-4}	3.19×10^{-3}	0.00704	4.77×10^{-4}	41
16	0.702×10^{-3}	0.00354	0.73×10^{-4}	6.23×10^{-3}	0.00347	14.18×10^{-4}	42
17	1.071×10^{-3}	0.00237	1.32×10^{-4}	0.68×10^{-3}	0.00343	0.88×10^{-4}	43
18	2.970×10^{-3}	0.00166	4.60×10^{-4}	1.10×10^{-3}	0.00244	1.28×10^{-4}	44
19	2.110×10^{-3}	0.00118	2.46×10^{-4}	2.09×10^{-3}	0.01052	3.56×10^{-4}	45
20	0.5×10^{-3}		8.95×10^{-4}	3.15×10^{-3}	0.00696	5.40×10^{-4}	46
21	"		0.77×10^{-4}	6.34×10^{-3}	0.00353	14.74×10^{-4}	47
22	0.33×10^{-3}		0.342×10^{-4}				48
23	0.322×10^{-3}		0.400×10^{-4}				49
24							50
25							51
26							52

Steel





1)

Important Papers on Curved Sheets

K. Marguerre: über die Behandlung von Stabilitätsproblemen
mit Hilfe der energetischen Methode
Z.A.M.M. 8 57-73 (1938)

✓ K. Marguerre, E. Treffitz Über die Tragfähigkeit
eines Plattenstreifens nach Überschreiten der Baulast
Z.A.M.M. 17: 85-100 (1937)

✓ K. v. Sanden, F. Tölke Über Stabilitätsprobleme dünner
kreiszyklindrischer Schalen Ing.-Arch 3 29-66
(1932)

W. Flügge Die Stabilität der Kreiszyklinderschale
Ing.-Arch 3 463-506 (1932)

✓ K. Marguerre Die mitttragende Breite der gedrückten Platte
Luftfahrtforschung (1937) 14 124

T. E. Schumacher zur Knickfestigkeit schwach gekrümmter
zyklindrischer Schalen Ing.-Arch 4 394-414
(1933)

✓ N. J. Hoff Instability of Monocoque Structures in Pure bending
Journal of Roy. Soc. (1938) 42: 291-305

- ✓ L. H. Donnell A New Theory for the Buckling of
Thin Cylinders Under Axial Compression &
Bending Trans. of ASME 79: 181-186 (1934) 2)
- R. V. Southwell On the General Theory of Elastic Stability
Trans. of Roy. Soc. of London 213: 187-244
- ✓ J. L. Taylor Stability of a Monocoque in Compression (1914)
R.M. 1679 (1936)
- ✓ H. Ebener, H. Köller Zur Berechnung des Kraftverlaufes
in versteiften Zylinderschalen
Zff. 14: 607-626 (1937) TM 866
- ✓ E. Schapitz, G. Krümling Belastungsversuche
mit einer versteiften Kreiszylinderschale bei
Krafteinleitung an einzelnen Punkten
Zff. 14: 593-606 (1937) TM 864
- ✓ H. Wagner Einiges über schalenförmiges Fluggew-
banteile Zff. 13: 281-292 (1936) TM 817
- ✓ O. S. Heck, H. Ebener Formeln und Berechnungsverfahren
für die Festigkeit von Platten- und ~~Stabtragwerk~~
Schalenkonstruktionen im Fluggewichtbau
Zff. 12: 211-222 (1935) TM 715

- ✓ A. J. Lutton Piffard Distortion of Thin Tubes under Flexure 3
RM 1465 (1932)
- ✓ W. Kaufmann Bemerkungen zur Elastizität dünnwandiger,
kreiszylindrischer Schalen oberhalb der
Proportionalitätsgrenze Ing - Arch 6 419-430
(1935)
- ✓ J. W. Geckeler Plastisches Knicken der Wandung
von Hohlzylindern und einige andere
Faltungerscheinungen an Schalen und Blechen
Zach. M. 8: 341-352 (1928)
- ✓ W. Kaufmann Plastisches Knicken dünnwandiger
Hohlzylinder infolge axialer Belastung
Ing - Arch. 6: 334-337 (1935)
- ✓ W. Kaufmann Über unelastisches Knicken rechteckiger Platten
Ing - Arch. 7: 156-165 (1936)
- ✓ A. Kromm, K. Marguerre Verhalten eines von Schub- und
Zugkräften beanspruchten Plattenstreifens oberhalb
der Beulgrenze Zff. 14: 627-639 (1937)
ZfM 820

R Barbé. Stabilität gleichmäßig gedrückter
Rechteckplatten mit Längs oder Quersteifen
Ing.-Arch. 8: 117-150 (1937) [GALLIT Trans.] 4'

A Pflüger. Stabilität dünner Kegelschalen
Ing.-Arch. 8: 151-172 (1937)

W A Wenzek. Die mitttragende Breite nach den Ausknicken
bei krummen Blechen Ziff. 15 340-344 (1935)

L Toppl. Eine neue elastische Materialkonstante
Ing.-Arch. 7: 229-236 (1936)

Section 2

Donnell's Equation

The Buckling of Circular Cylindrical Shell

Dove's Equation (Cong. of Applied Mech.)

7

$$\frac{Et^2}{12(1-\mu^2)} \left\{ r^2 \nabla^4 w + \frac{2}{r^2} \frac{\partial^4 w}{\partial r^2 \partial \theta^2} + \frac{1}{r^4} \frac{\partial^4 w}{\partial \theta^4} \right\} + E \frac{\partial^4 w}{\partial x^4} = -\sigma \frac{\partial^2}{\partial x^2} \left\{ r^2 \nabla^2 w - \frac{1}{r^2} \frac{\partial^2 w}{\partial \theta^2} \right\}$$

Put $w = \sin nt \sin \frac{2\pi x}{l}$

Wave length in axial direction = l , in circumferential direction = $\frac{2\pi}{n} r = m$, $\frac{2\pi}{m} = \frac{n}{r}$

$$\frac{t^2}{12(1-\mu^2)} \left[r^2 \left\{ \left(\frac{2\pi}{l} \right)^4 + \left(\frac{n}{r} \right)^4 \right\} - 2 \left(\frac{2\pi}{l} \right)^2 \left(\frac{n}{r} \right)^2 + \frac{1}{r^2} \left(\frac{n}{r} \right)^4 \right] + \left(\frac{2\pi}{l} \right)^4 = \frac{\sigma}{E} \left[r^2 \left(\frac{2\pi}{l} \right)^2 \left\{ \left(\frac{2\pi}{l} \right)^2 + \left(\frac{n}{r} \right)^2 \right\} + \frac{1}{r^2} \left(\frac{n}{r} \right)^2 \right]$$

$$\frac{t^2}{12(1-\mu^2)} \left[r^2 \left\{ \left(\frac{2\pi}{l} \right)^4 + \left(\frac{n}{r} \right)^4 \right\} - 2 \left(\frac{2\pi}{l} \right)^2 \left(\frac{n}{r} \right)^2 + \frac{1}{r^2} \left(\frac{n}{r} \right)^4 \right] + \left(\frac{2\pi}{l} \right)^4$$

$$= \frac{\sigma}{E} \left[r^2 \left(\frac{2\pi}{l} \right)^2 \left\{ \left(\frac{2\pi}{l} \right)^2 + \left(\frac{n}{r} \right)^2 \right\} + \frac{1}{r^2} \left(\frac{n}{r} \right)^2 \right]$$

Put $r^2 = R^2 \frac{l^2}{12(1-\mu^2)}$ $\left(\frac{2\pi}{l} \right)^2 = \frac{\alpha}{\frac{l^2}{12(1-\mu^2)}}$

$$\left(\frac{n}{m} \right)^2 = \frac{\beta}{\frac{l^2}{12(1-\mu^2)}}$$

$$R^2 (\alpha + \beta)^4 - 2\beta^3 + \frac{\beta^2}{R^2} + \alpha^2 = \frac{\sigma}{E} \left[R^2 \alpha (\alpha + \beta)^2 + \alpha \beta \right]$$

$$\therefore \frac{\sigma}{E} = \frac{R^2 (\alpha + \beta)^4 - 2\beta^3 + \frac{\beta^2}{R^2} + \alpha^2}{R^2 \alpha (\alpha + \beta)^2 + \alpha \beta}$$

$$\text{Let } \left(\frac{m}{L}\right)^2 = \lambda = \frac{\alpha}{\beta} \quad \text{or} \quad \alpha = \lambda\beta \quad 2)$$

$$\frac{\sigma}{E} = \frac{R^2 \beta^2 (1+\lambda)^4 - 2\beta + \frac{1}{R^2} + \lambda^2}{R^2 \lambda \beta (1+\lambda)^2 + \lambda}$$

$$\left\{ R^2 \lambda \beta (1+\lambda)^2 + \lambda \right\} \left\{ 2R^2 (1+\lambda)^2 \beta - 2 \right\} \\ - \left\{ R^2 \beta^2 (1+\lambda)^4 - 2\beta + \frac{1}{R^2} + \lambda^2 \right\} R^2 \lambda (1+\lambda)^2 = 0$$

$$R^4 \lambda (1+\lambda)^6 \beta^2 + 2R^2 \lambda (1+\lambda)^4 \beta - \left\{ \lambda (1+\lambda)^2 + 2\lambda + R^2 \lambda^3 (1+\lambda)^2 \right\} = 0$$

$$\lambda \sim 1, \quad R \gg 1,$$

$$R^2 (1+\lambda)^4 \beta^2 + 2(1+\lambda)^2 \beta - \lambda^2 = 0.$$

$$\beta^2 + \frac{2}{R^2 (1+\lambda)^2} \beta - \frac{\lambda^2}{R^2 (1+\lambda)^4} = 0$$

$$\beta = -\frac{1}{R^2 (1+\lambda)^2} + \sqrt{\frac{1}{R^4 (1+\lambda)^4} + \frac{\lambda^2}{R^2 (1+\lambda)^4}}$$

$$= \frac{1}{R^2 (1+\lambda)^2} \left[\sqrt{1 + \lambda^2 R^2} - 1 \right]$$

$$\approx \frac{\lambda}{R (1+\lambda)^2}$$

$$\begin{aligned}\frac{\sigma}{E} &= \frac{2\lambda^2 + \frac{1}{R^2} - 2[(1+\lambda)^2 + 1]\beta}{R^2\lambda(1+\lambda)^2\beta + \lambda} \\ &= \frac{2\lambda^2 + \frac{1}{R^2} - \frac{2\lambda[(1+\lambda)^2 + 1]}{R(1+\lambda)^2}}{R\lambda^2 + \lambda} \approx \frac{2}{R}\end{aligned}$$

$$\frac{\sigma}{E} = \frac{2}{\left(\frac{t}{R}\right)\sqrt{12(1-\mu^2)}}$$

$$\sigma = \frac{E}{\sqrt{3(1-\mu^2)}} \left(\frac{t}{R}\right) \quad \underline{\text{Working Ex.}}$$

$$R^2(1+\lambda)^6 \beta^2 + 2R^2(1+\lambda)^3 \beta - \left\{ (3+2\lambda+\lambda^2) + R^2\lambda^2(1+\lambda)^2 \right\} = 0 \quad 4)$$

$$\beta^2 + \frac{2}{R^2(1+\lambda)^3} \beta - \left\{ \frac{(3+2\lambda+\lambda^2)}{R^2(1+\lambda)^6} + \frac{\lambda^2}{R^2(1+\lambda)^4} \right\} = 0$$

$$\beta = -\frac{1}{R^2(1+\lambda)^3} + \sqrt{\frac{1}{R^4(1+\lambda)^6} + \frac{\lambda^2}{R^2(1+\lambda)^4} + \frac{3+2\lambda+\lambda^2}{R^4(1+\lambda)^6}}$$

$$= \frac{1}{R^2(1+\lambda)^3} \left[\sqrt{1 + \lambda^2 R^2 + \frac{3+2\lambda+\lambda^2}{(1+\lambda)^2}} - 1 \right]$$

$$\frac{\sigma}{E} = \frac{R^2(1+\lambda)^2 \beta^2 - 2\beta + \frac{1}{R^2} + \lambda^2}{R^2\lambda(1+\lambda)^2 \beta + \lambda}$$

$$= \frac{\frac{(3+2\lambda+\lambda^2)}{R^2(1+\lambda)^2} + \lambda^2 - 2\left[(1+\lambda)^2 + 1\right]\beta + \frac{1}{R^2}}{R^2\lambda(1+\lambda)^2 \beta + \lambda}$$

$$\frac{\sigma}{E} = \frac{\frac{(4+4\lambda+2\lambda^2)}{R^2(1+\lambda)^2} + \lambda^2 - 2(2+2\lambda+\lambda^2)\beta}{\lambda[R^2(1+\lambda)^2 \beta + 1]}$$

$$\frac{\sigma}{E} = \frac{\frac{(4+4\lambda+2\lambda^2)}{R^2(1+\lambda)^2} + 2\lambda^2 - 2(2+\lambda+\lambda^2)\beta}{\lambda \sqrt{1+\lambda^2 R^2 + \frac{3+2\lambda+\lambda^2}{(1+\lambda)^2}}}$$

for $\lambda = 1$

$$\frac{\sigma}{E} = \frac{\frac{9}{4} \frac{1}{R^2} + 2 - 10 \left\{ \frac{\sqrt{1+R^2 + \frac{5}{4}}}{4R^2} - 1 \right\}}{\sqrt{1+R^2 + \frac{5}{4}}}$$

$$\frac{\sigma}{E} = \frac{\frac{1}{4R^2} [19 - 10 \sqrt{1+R^2 + \frac{5}{4}}] + 2}{\sqrt{1+R^2 + \frac{5}{4}}} = \frac{\frac{19}{4R^2} + 2}{\sqrt{1+R^2 + \frac{5}{4}}} - \frac{5}{2R^2}$$

$$\boxed{\frac{\sigma}{E} = \frac{2 + \frac{19}{4R^2}}{\sqrt{R^2 + \frac{9}{4}}} - \frac{5}{2R^2}}$$

$\lambda = 100$

$$\frac{\sigma}{E} = \frac{\frac{10403}{R^2 10201} + 20000 - 2 \times 10202 \frac{\sqrt{1+10000R^2 + \frac{10202}{10201}} - 1}{R^2 10201}}{100 \sqrt{1+10000R^2 + \frac{10202}{10201}}}$$

$$\begin{aligned} &\approx \frac{\frac{1.020}{R^2} + 20000 - \frac{200}{R}}{10000R} = \frac{2}{R} - \frac{0.02}{R^2} + \frac{0.00102}{R^3} \\ &= \frac{2}{R} \left(1 - \frac{0.01}{R} + \frac{0.00051}{R^2} \right) \end{aligned}$$

6)

$$\lambda = 0.1$$

$$\frac{\sigma}{E} = \frac{\frac{4.42}{1.21} \frac{1}{R^2} + 0.2 - 4.42 \frac{\sqrt{1 + 0.01R^2 + \frac{2.21}{1.21}} - 1}{1.21 R^2}}{0.1 \sqrt{1 + 0.01R^2 + \frac{2.21}{1.21}}}$$

$$= \frac{\frac{3.6540}{R^2} + 0.2 - \frac{3.6529}{R^2} \left\{ \sqrt{3.652 + 0.01R^2} - 1 \right\}}{0.1 \sqrt{2.8164 + 0.01R^2}}$$

$$\text{If } R = 100$$

$$\frac{\sigma}{E} = \frac{1}{R} \left[\frac{0.036540 + 2 - 0.036529 \times 10.80}{0.1 \times 11.20} \right] \quad 0.822$$

$$R = 300$$

$$\frac{\sigma}{E} = \frac{1}{R} \left[\frac{0.01218 + 2 - 0.012176 \times 27.048}{0.1 \times 30.247} \right] \quad 0.209$$

$$R = 1000$$

$$\frac{\sigma}{E} = \frac{1}{R} \left[\frac{0.0028264 + 2 - 0.002829 \times 100}{0.1 \times 101} \right] = \quad 0.595$$

$$R = 10$$

$$\frac{\sigma}{E} = \frac{1}{R} \left[\frac{0.36540 + 0.2 - 0.36529 \times 1.164}{0.1 \times 2.964} \right] \quad 0.314$$

$$L = 0.01$$

c)

$$\frac{\sigma}{E} = \frac{\frac{30402}{10201} \frac{1}{R^2} + 0.0002 - \frac{4.2432}{10201 R^2} \left[\sqrt{1 + \frac{30402}{10201} + 0.0001 R^2} - 1 \right]}{0.01 \sqrt{1 + \frac{30402}{10201} + 0.0001 R^2}}$$

for $R = 100$

$$\frac{\sigma}{E} = \frac{1}{R} \left[\frac{0.036548 + 0.0002 - \frac{4.2432}{10201} \times 0.51226}{0.01 \times 9.3261} \right]$$

$$= \frac{1}{R} \left[\frac{0.010391}{0.01 \times 1.9971} \right] = \frac{1}{R} \times 0.2604, \quad k = 0.1578$$

$R = 300$

$$\frac{\sigma}{E} = \frac{1}{R} \left[\frac{0.01217 \times 1 + 0.0002 - \frac{4.2432}{10201} \times \frac{26612}{300}}{0.01 \times 3.6060} \right]$$

$$= \frac{1}{R} \times \frac{0.01146}{0.01 \times 3.6060} \quad 0.3175$$

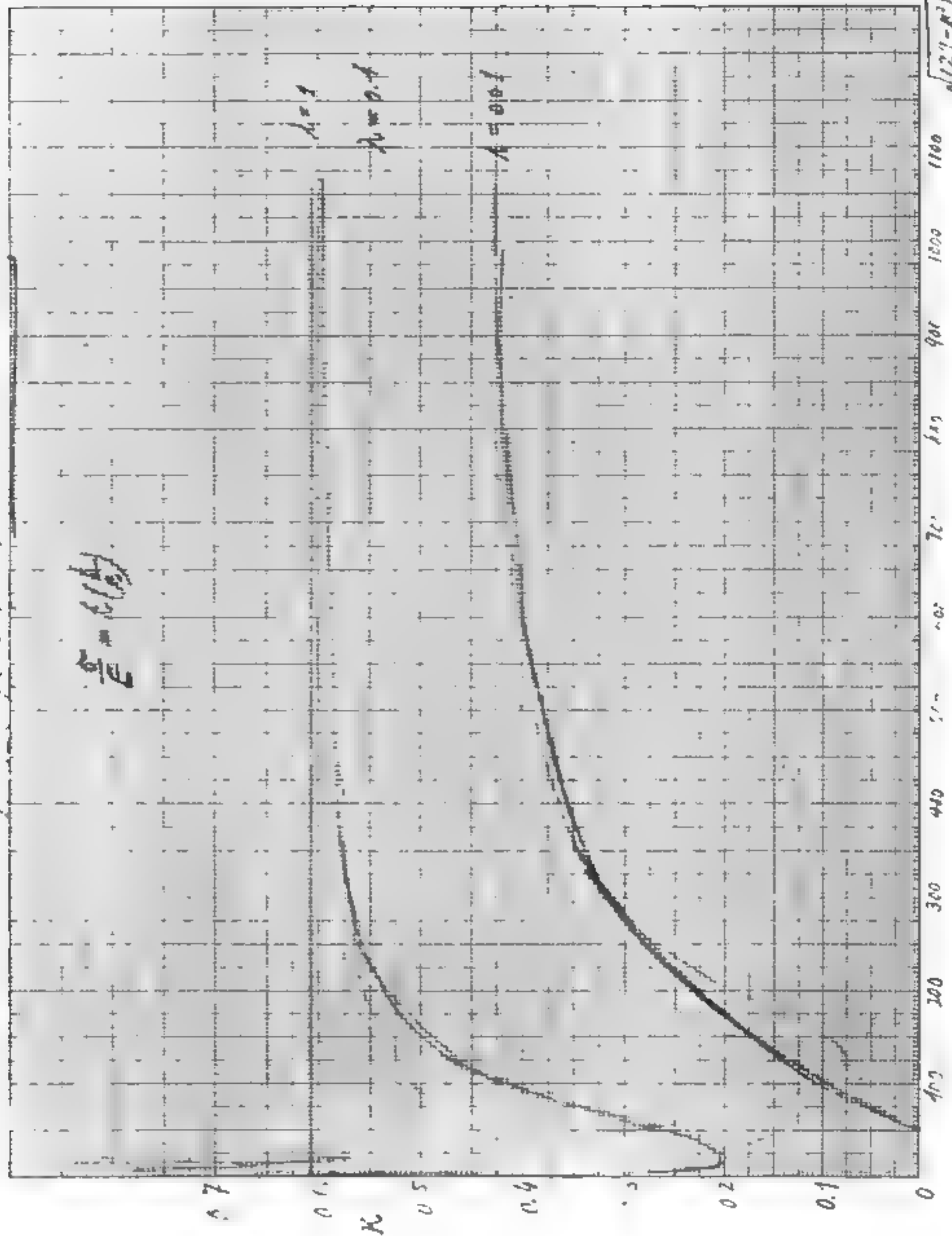
$R = 1000$

$$\frac{\sigma}{E} = \frac{1}{R} \left[\frac{0.003923 + 0.0002 - \frac{4.2432}{10201} \times \frac{26612}{1000}}{0.01 \times 11.49} \right] = 0.426$$

$R = 50$

$$\frac{\sigma}{E} = \frac{1}{R} \left[\frac{0.07308 \times 0.1 - \frac{4.2432}{10201} \times \frac{1.052}{50}}{0.01 \times 2.052} \right] \sim 0$$

1.57 1000 1000 1000



$$\frac{d}{dt} = \frac{d}{dt} \left(\frac{1}{2} \right)$$

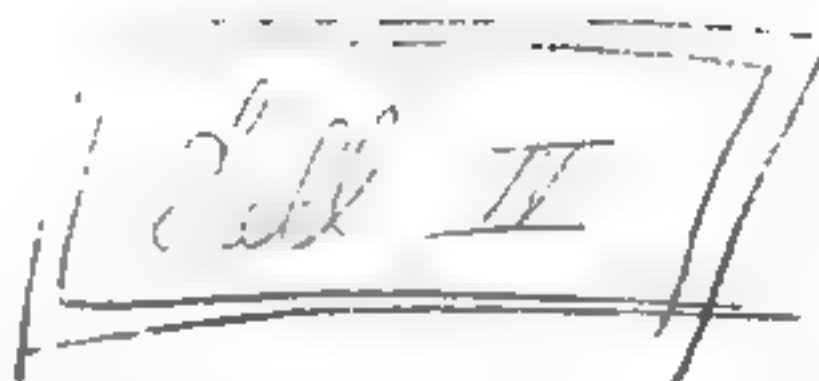
Section 3

Preliminary Calculation of Circular Cylinder (I)

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Preliminary Calculation of Circular Cylinder

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PART (2)

The Increase in Strain Energy Due to the Presence of a Circular Hole

We have from Southwell's "Elasticity", p 386, the following relations

$$\hat{r}_r = \frac{1}{2} T \left(1 - \frac{a^2}{r^2}\right) \left[1 + (1 - 3\frac{a^2}{r^2}) \cos 2\theta\right]$$

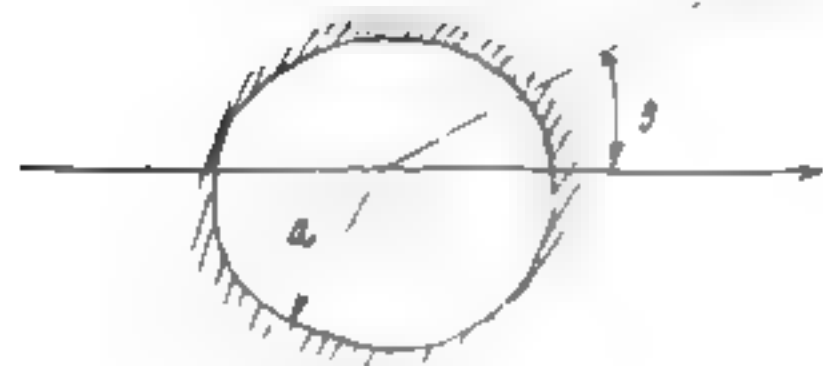
$$\hat{r}_\theta = \frac{1}{2} T \left[1 + \frac{a^2}{r^2} - (1 + 3\frac{a^2}{r^2}) \cos 2\theta\right]$$

$$\hat{r}_\theta = -\frac{1}{2} T \left(1 - \frac{a^2}{r^2}\right) (1 + 3\frac{a^2}{r^2}) \sin 2\theta$$

$$\hat{r}_{r_0} = \frac{1}{2} T (1 + \cos 2\theta)$$

$$\hat{r}_{\theta_0} = \frac{1}{2} T (1 - \cos 2\theta)$$

$$\hat{r}_{\theta_0} = -\frac{1}{2} T \sin 2\theta$$



If we assume that $\mu = 0$, i.e., vanishing Poisson's ratio, then
In case of plane stress,

$$\text{strain energy per unit area} = \frac{1}{E} \left\{ \frac{\hat{r}_r^2 + \hat{r}_\theta^2}{2} - \nu \hat{r}_r \hat{r}_\theta + (1+\nu) \hat{r}_{\theta_0}^2 \right\}$$

$$\begin{aligned}
& \text{Thus } \left[\frac{\hat{a}\hat{a}^2 + \hat{b}\hat{b}^2}{2} - v \hat{b}\hat{a} + (1+v) \hat{b}\hat{a}^2 \right] \left[\frac{\hat{a}\hat{a}_0^2 + \hat{b}\hat{b}_0^2}{2} - v \hat{b}_0\hat{a}_0 + (1+v) \hat{b}_0\hat{a}_0^2 \right] \quad \underline{263} \\
&= \frac{1}{4} T^2 \left\{ \left[\left(1 - \frac{a^2}{\lambda^2}\right)^2 \left[1 + \left(1 - 3\frac{a^2}{\lambda^2}\right) \cos 2\theta \right]^2 + \left[1 + \frac{a^2}{\lambda^2} - \left(1 + \frac{3a^2}{\lambda^2}\right) \cos 2\theta \right]^2 \right. \right. \\
&\quad \left. \left. - (1 + \cos 2\theta)^2 - (1 - \cos 2\theta)^2 \right] \right. \\
&\quad \left. - v \left[\left(1 - \frac{a^2}{\lambda^2}\right) \left[1 + \left(1 - 3\frac{a^2}{\lambda^2}\right) \cos 2\theta \right] \left[1 + \frac{a^2}{\lambda^2} - \left(1 + \frac{3a^2}{\lambda^2}\right) \cos 2\theta \right] - (1 + \cos 2\theta)(1 - \cos 2\theta) \right] \right. \\
&\quad \left. + (1+v) \left[\left(1 - \frac{a^2}{\lambda^2}\right) \left(1 + 3\frac{a^2}{\lambda^2}\right) \sin^2 \theta - \sin^2 \theta \right] \right\} \\
&= \left(\frac{T}{2}\right)^2 \left\{ \frac{1}{2} \left[\frac{a^2}{\lambda^2} \left(\frac{a^2}{\lambda^2} - 2 \right) \left[1 + \left(1 - 3\frac{a^2}{\lambda^2}\right) \cos 2\theta \right]^2 + 3\frac{a^2}{\lambda^2} \cos 2\theta \left(3\frac{a^2}{\lambda^2} \cos 2\theta - 2(1 + \cos 2\theta) \right) \right. \right. \\
&\quad \left. \left. + \frac{a^2}{\lambda^2} \left(1 - \frac{3a^2}{\lambda^2} \cos 2\theta \right) \left[\frac{a^2}{\lambda^2} \left(1 - \frac{3a^2}{\lambda^2} \right) \cos 2\theta + 2(1 - \cos 2\theta) \right] \right] \right. \\
&\quad \left. - v \left[-\frac{2}{\lambda^2} \left\{ (1 + \cos 2\theta) - 3\frac{a^2}{\lambda^2} \cos 2\theta \right\} \left(1 - \cos 2\theta \right) + \frac{2}{\lambda^2} \left(1 - \frac{3a^2}{\lambda^2} \right) (1 + \cos 2\theta) \right. \right. \\
&\quad \left. \left. + \frac{2}{\lambda^2} \cos 2\theta \left\{ (1 + \cos 2\theta) \left(1 - \frac{3a^2}{\lambda^2} \right) - 3(1 - \cos 2\theta) \right\} - 3 \left(1 - \frac{3a^2}{\lambda^2} \right) \frac{a^2}{\lambda^2} \cos^2 2\theta \right] \right. \\
&\quad \left. + (1+v) \left[\sin^2 \theta \left\{ \left(1 - \frac{a^2}{\lambda^2} \right) \left(1 + \frac{3a^2}{\lambda^2} \right) + 1 \right\} \frac{a^2}{\lambda^2} \left(2 - 3\frac{a^2}{\lambda^2} \right) \right] \right\} = F(\lambda, \theta) \left(\frac{T}{2}\right)^2
\end{aligned}$$

$$\begin{aligned}
 F(r, \theta) = & \frac{1}{2} \left\{ \frac{a^2}{r^2} \left(\frac{r^2}{a^2} - 2 \right) \left[1 + \left(1 - 3 \frac{a^2}{r^2} \right) \cos^2 \theta \right] + 3 \frac{a^2}{r^2} \left[3 \frac{a^2}{r^2} \cos^2 \theta \right. \right. \\
 & \left. \left. + \frac{a^2}{r^2} \left(1 - \frac{3a^2}{r^2} \right) \cos \theta + 2 \left(1 - \frac{a^2}{r^2} \right) \right] - \frac{3a^2}{r^2} \left[\frac{a^2}{r^2} \left(1 - \frac{3a^2}{r^2} \right) \cos^2 \theta + 2 \left(1 - \frac{a^2}{r^2} \right) \cos \theta \right] \right. \\
 & \left. - \pi \left\{ -\frac{a^2}{r^2} \left[\sin^2 \theta - 3 \frac{a^2}{r^2} (\cos \theta - \cos^3 \theta) + \frac{a^2}{r^2} \left(1 - \frac{3a^2}{r^2} \right) (\cos \theta + \cos^3 \theta) \right] \right. \right. \\
 & \left. \left. + \frac{a^2}{r^2} \left[\left(1 - \frac{3a^2}{r^2} \right) (\cos \theta + \cos^3 \theta) - 3 (\cos \theta - \cos^3 \theta) \right] - 3 \left(1 - \frac{3a^2}{r^2} \right) \frac{a^4}{r^4} \cos^3 \theta \right\} \right. \\
 & \left. + (1+r) \left\{ \sin^2 \theta - \frac{a^2}{r^2} \left(2 - 3 \frac{a^2}{r^2} \right) \left[\left(1 - \frac{a^2}{r^2} \right) \left(1 + \frac{3a^2}{r^2} \right) + 1 \right] \right\} \right\}
 \end{aligned}$$

$$\begin{aligned}
 \therefore \int_0^\pi F(r, \theta) d\theta = & \frac{\pi}{2} \left\{ \frac{a^2}{r^2} \left(\frac{a^2}{r^2} - 2 \right) \left[2 + \left(1 - 3 \frac{a^2}{r^2} \right) \right] + 3 \frac{a^2}{r^2} \left[3 \frac{a^2}{r^2} - 2 \right] \right. \\
 & \left. + \frac{a^2}{r^2} \cdot 4 - \frac{3a^4}{r^4} \left[\frac{a^2}{r^2} \left(1 - \frac{3a^2}{r^2} \right) - 2 \right] \right\} \\
 & - \pi \left\{ -\frac{a^2}{r^2} \left[1 + 3 \frac{a^2}{r^2} + \frac{a^2}{r^2} \left(1 - \frac{3a^2}{r^2} \right) \right] + \frac{a^2}{r^2} \left[\left(1 - \frac{3a^2}{r^2} \right) + 3 \right] - 3 \left(1 - \frac{3a^2}{r^2} \right) \frac{a^4}{r^4} \right\} \\
 & + \pi (1+r) \frac{a^2}{r^2} \left(2 - 3 \frac{a^2}{r^2} \right) \left[2 + 2 \frac{a^2}{r^2} - \frac{3a^4}{r^4} \right] \quad \text{Putting } \frac{a}{r} = \frac{1}{k}
 \end{aligned}$$

$$\begin{aligned}
 = & \frac{\pi}{2} \left\{ \frac{1}{k^2} \left(\frac{1}{k^2} - 2 \right) \left(3 - 6 \frac{1}{k^2} + 9 \frac{1}{k^4} \right) + 3 \frac{1}{k^2} \left(3 \frac{1}{k^2} - 2 \right) + 4 \frac{1}{k^2} - 3 \frac{1}{k^4} \left[\frac{1}{k^2} - \frac{3}{k^4} - 2 \right] \right\} \\
 & - \pi \left\{ -\frac{1}{k^2} \left[1 + 4 \frac{1}{k^2} - \frac{3}{k^4} \right] + \frac{1}{k^2} \left[4 - \frac{3}{k^2} \right] - 3 \frac{1}{k^2} \left(1 - \frac{3}{k^2} \right) \right\} \\
 & + \pi (1+r) \frac{1}{k^2} \left(2 - 3 \frac{1}{k^2} \right) \left(2 + \frac{2}{k^2} - \frac{3}{k^4} \right)
 \end{aligned}$$

If for simplicity $N=0$, then

$$\begin{aligned} \frac{R^4}{\pi} \int_0^{2\pi} F(R, \theta) d\theta &= \left[\frac{1}{2} \left(3 - \frac{6}{R^2} + 12 + \frac{9}{R^4} - 18 + 9 - \frac{3}{R^2} + \frac{9}{R^4} + 6 \right) \right. \\ &\quad \left. + \left(-6 + 4 - \frac{6}{R^2} - \frac{6}{R^2} + \frac{9}{R^4} \right) \right] \\ &= \left[4 - \frac{33}{2} \frac{1}{R^2} + 18 \frac{1}{R^4} \right] \end{aligned}$$

Energy increase due to the presence of hole of radius "a"

$$\begin{aligned} &= \int_0^{\infty} \int_0^{2\pi} \left(\frac{1}{2} - \frac{t}{E} \right) F(R, \theta) R dR d\theta \\ &= \frac{a^2 t}{E} \left(\frac{\sigma}{2} \right)^2 \int_0^{\infty} F(R, \theta) R dR \\ &= \frac{a^2 \pi t}{E} \left(\frac{\sigma}{2} \right)^2 \int_0^{\infty} \left[4 - \frac{33}{2} \frac{1}{R^2} + \frac{18}{R^4} \right] R dR \\ &= \frac{a^2 \pi t}{E} \left(\frac{\sigma}{2} \right)^2 \left[2 - \frac{33}{8} + 3 \right] = \underbrace{\left[\frac{a^2 \pi t}{E} \left(\frac{\sigma}{2} \right)^2 \right]}_{\text{}} \underbrace{\frac{7}{8}}_{\text{}} = \frac{a^2 \pi t \sigma^2}{E} \frac{7}{32} \end{aligned}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \frac{1}{E} \hat{\lambda}$$

$$\frac{\mu}{2} + \frac{1}{\lambda} \frac{\partial \mathcal{L}}{\partial \xi} = \frac{1}{E} \hat{\theta}$$

$$\frac{1}{\lambda} \frac{\partial \mu}{\partial \theta} + \frac{\partial \theta}{\partial \lambda} - \frac{\partial}{\partial \lambda} = \frac{\partial}{E} \hat{\theta}$$

$$\frac{\partial \mu}{\partial \lambda} = \frac{\sigma}{2E} \left(1 - \frac{a^2}{\lambda^2} \right) \left[1 + \left(1 - \frac{3a^2}{\lambda^2} \right) \cos 2\theta \right]$$

$$= \frac{\sigma}{2E} \left[\left(1 - \frac{a^2}{\lambda^2} \right) + \left(1 - \frac{4a^2}{\lambda^2} + \frac{3a^4}{\lambda^4} \right) \cos 2\theta \right]$$

$$\mu = \frac{\sigma}{2E} \left[\left(\lambda + \frac{a^2}{\lambda} \right) + \left(\lambda + \frac{4a^2}{\lambda} - \frac{a^4}{\lambda^3} \right) \cos 2\theta + F(\theta) \right]$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = \frac{1}{E} \lambda \hat{\theta} - \mu$$

$$= \frac{\sigma}{2E} \left\{ \lambda + \frac{a^2}{\lambda} - \left(\lambda + \frac{3a^4}{\lambda^3} \right) \cos 2\theta - \left(\lambda + \frac{a^2}{\lambda} \right) - \left(\lambda + \frac{4a^2}{\lambda} - \frac{a^4}{\lambda^3} \right) \cos 2\theta - F(\theta) \right\}$$

$$= \frac{\sigma}{2E} \left\{ -2 \left(\lambda + \frac{3a^2}{\lambda} + \frac{a^4}{\lambda^3} \right) \cos 2\theta - F(\theta) \right\}$$

$$v = \frac{\sigma}{2E} \left\{ - \left(\lambda + \frac{2a^2}{\lambda} + \frac{a^4}{\lambda^3} \right) \sin 2\theta - \int F(\theta) + E(\lambda) \right\}$$

Check.

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$$\begin{aligned} & \frac{\sigma}{2E} \left[-2 \left(1 + \frac{4a^2}{r^2} - \frac{a^4}{r^4} \right) \sin 2\theta + \frac{F'(b)}{r} - \left(1 - \frac{2a^2}{r^2} - \frac{3a^4}{r^4} \right) \sin 2\theta + G'(r) \right. \\ & \quad \left. + \left(1 + \frac{3a^2}{r^2} + \frac{a^4}{r^4} \right) \sin 2\theta + \frac{F(b)}{r} + \frac{G(r)}{r} \right] \\ &= \frac{\sigma}{E} \left[\left(\frac{2a^2}{r^2} + \frac{3a^4}{r^4} - 1 - \frac{4a^2}{r^2} + \frac{a^4}{r^4} \right) \sin 2\theta + \frac{F'(b) + G'(r) + F(b)}{r} + G'(r) \right] \\ &= -\frac{\sigma}{E} \left(1 + \frac{2a^2}{r^2} - \frac{3a^4}{r^4} \right) \sin 2\theta + \frac{\sigma}{E} \left[\right] \end{aligned}$$

$$\therefore \frac{v}{r} = -\frac{\sigma}{2E} \left(1 + \frac{a^2}{r^2} \right)^2 \sin 2\theta$$

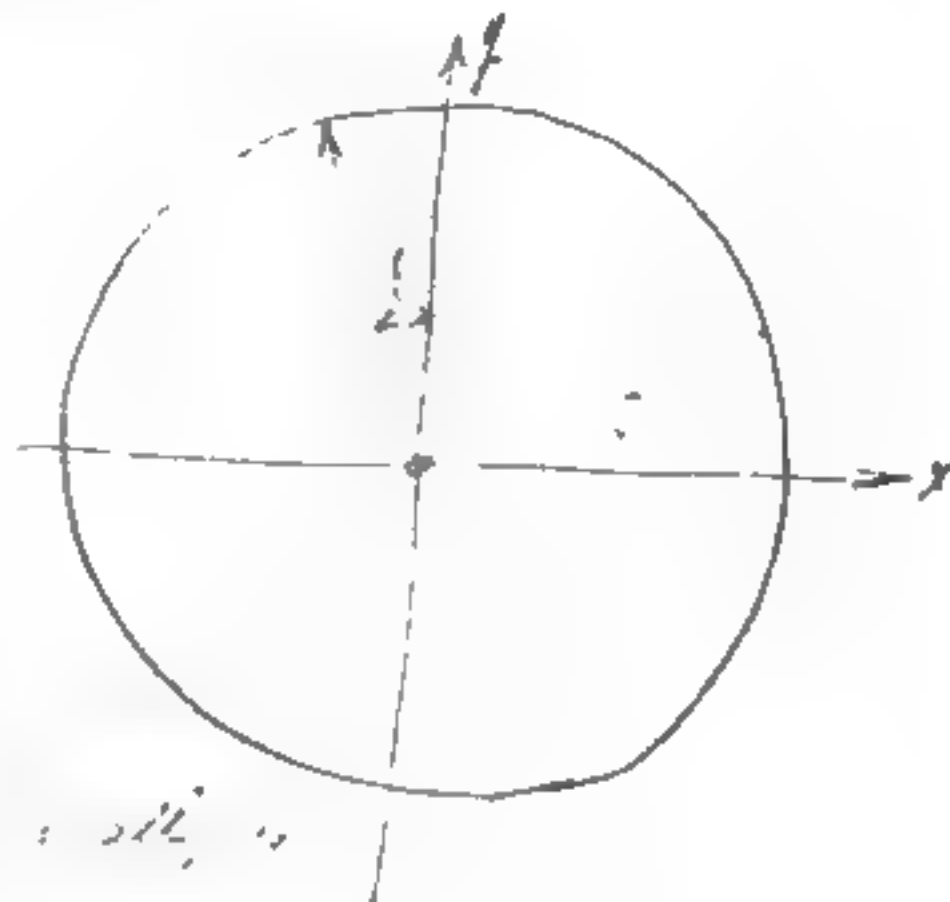
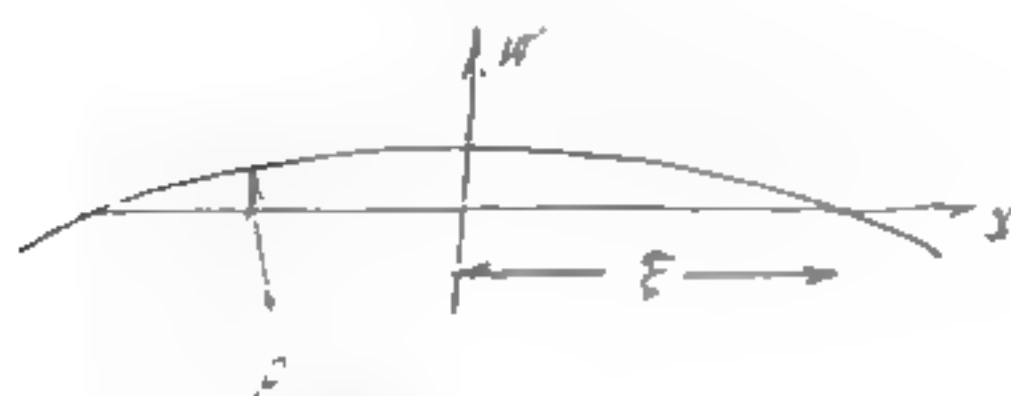
$$\frac{u}{r} = \frac{\sigma}{2E} \left[\left(1 + \frac{a^2}{r^2} \right) + \left(1 + \frac{4a^2}{r^2} - \frac{a^4}{r^4} \right) \cos 2\theta \right]$$

At $r = a$,

$$\begin{aligned} \frac{v}{a} &= -\frac{\sigma}{2E} 4 \sin 2\theta = -\frac{2\sigma}{E} \sin 2\theta \\ \frac{u}{a} &= \frac{\sigma}{2E} (2 + 4 \cos 2\theta) = \frac{\sigma}{E} (1 + 2 \cos 2\theta) \end{aligned}$$

Calculation of the Strain Energy of a Cylindrical Shell
of Circular Cross Section
with Given Displacement

1) The actual displacement:



As a part of the shell, the
the actual displacement of the shell is

$$\left(\frac{w}{R}\right)_0 = \frac{\left(\frac{x}{R}\right)^2 - \left(\frac{y}{R}\right)^2}{2} = \frac{\left(\frac{x}{R}\right)^2 - \left(\frac{y}{R}\right)^2}{2}$$

After substituting

$$\left(\frac{w}{R}\right) = \frac{\frac{x^2}{R^2} - \frac{y^2}{R^2}}{2} + f \left[\frac{\left(\frac{x}{R}\right)^2 - \left(\frac{y}{R}\right)^2}{2} \right]$$

$$\left(\frac{w}{R}\right) = \frac{1}{2} \left\{ \frac{\left(\frac{x}{R}\right)^2 - \left(\frac{y}{R}\right)^2}{2} + f \left[\frac{\left(\frac{x}{R}\right)^2 - \left(\frac{y}{R}\right)^2}{2} \right] \right\}$$

Changing to polar coordinates

$$\left(\frac{w}{R}\right)_0 = \frac{\left(\frac{x}{R}\right)^2 - \left(\frac{y}{R}\right)^2 \sin^2 \theta}{2}$$

$$\left(\frac{w}{R}\right) = \frac{\left(\frac{x}{R}\right)^2 - \left(\frac{y}{R}\right)^2 \sin^2 \theta}{2} + f \left[\frac{\left(\frac{x}{R}\right)^2 - \left(\frac{y}{R}\right)^2 \sin^2 \theta}{2} \right] \quad \left. \vphantom{\frac{\left(\frac{x}{R}\right)^2 - \left(\frac{y}{R}\right)^2 \sin^2 \theta}{2}} \right\} \text{Form (1)}$$

To avoid any bending moment at the edges, we approximate 268
 the circular arc with a sine wave. The amplitude of the
 sine wave is given by

$$R(1 - \cos \beta) = 2R \sin^2 \frac{\beta}{2} = R \frac{\beta^2}{2} \approx R \frac{(\frac{a}{R})^2}{2} \approx \underline{\underline{\frac{a^2}{2R}}}$$

$$\left(\frac{w}{R}\right)_0 = \frac{a^2}{2R} \cos \frac{\pi x}{2a}$$

$$\left(\frac{w}{R}\right) = \frac{\left(\frac{a}{R}\right)^2 - \left(\frac{a}{R}\right)^2 \sin^2 \theta}{2} - f \cos \frac{\pi r}{2a}$$

2) The system of equations of stresses in the middle plane:

$$\left. \begin{aligned} \frac{\partial \sigma_r}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\tau_{r\theta}}{r} - \sigma_\theta &= 0 \\ \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \epsilon_{r\theta}}{\partial r} + \frac{2\tau_{r\theta}}{r} &= 0 \end{aligned} \right\}$$

These equations are satisfied by introducing the stress function φ

$$\sigma_r = \frac{1}{r} \frac{\partial \varphi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \varphi}{\partial \theta^2}$$

$$\sigma_\theta = \frac{\partial^2 \varphi}{\partial r^2}$$

$$\tau_{r\theta} = \frac{1}{r^2} \frac{\partial \varphi}{\partial \theta} - \frac{1}{r} \frac{\partial^2 \varphi}{\partial r \partial \theta} = - \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \varphi}{\partial \theta} \right)$$

$$\sigma_r = E \left\{ \frac{\partial u}{\partial r} + \frac{1}{2} \left(\frac{\partial w}{\partial r} \right)^2 - \frac{1}{2} \left(\frac{\partial w}{\partial \theta} \right)^2 \right\}$$

$$\sigma_\theta = E \left\{ \frac{u}{r} + \frac{\partial v}{r \partial \theta} + \frac{1}{2} \left(\frac{\partial w}{\partial r} \right)^2 - \frac{1}{2} \left(\frac{\partial w}{\partial \theta} \right)^2 \right\}$$

$$\tau_{r\theta} = \frac{E}{2} \left\{ \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} + \frac{1}{r} \frac{\partial v}{\partial r} \frac{\partial w}{\partial \theta} - \frac{1}{r} \frac{\partial w}{\partial r} \frac{\partial v}{\partial \theta} \right\}$$

$$\frac{\partial^2 w}{\partial r^2} = \cos^2 \theta \frac{\partial^2 w}{\partial r^2} - \frac{\sin^2 \theta}{r} \frac{\partial^2 w}{\partial r \partial \theta} + \frac{\sin^2 \theta}{r} \frac{\partial w}{\partial r} + \frac{\sin 2\theta}{r^2} \frac{\partial w}{\partial \theta} + \frac{\sin^2 \theta}{r^2} \frac{\partial^2 w}{\partial \theta^2}$$

$$\frac{\partial^2 w}{\partial \theta^2} = \sin^2 \theta \frac{\partial^2 w}{\partial r^2} + \frac{\sin 2\theta}{r} \frac{\partial^2 w}{\partial r \partial \theta} + \frac{\cos^2 \theta}{r} \frac{\partial w}{\partial r} - \frac{\sin 2\theta}{r^2} \frac{\partial w}{\partial \theta} + \frac{\cos^2 \theta}{r^2} \frac{\partial^2 w}{\partial \theta^2}$$

$$\frac{\partial}{\partial \theta} = \frac{\partial r}{\partial \theta} \frac{\partial}{\partial r} + \frac{\partial \theta}{\partial \theta} \frac{\partial}{\partial \theta} = \sin \theta \frac{\partial}{\partial r} + \frac{\cos \theta}{r} \frac{\partial}{\partial \theta}$$

$$\frac{\partial^2 w}{\partial r \partial \theta} = \left(\sin \theta \frac{\partial}{\partial r} + \frac{\cos \theta}{r} \frac{\partial}{\partial \theta} \right) \left(\frac{\partial w}{\partial r} \cos \theta - \frac{1}{r} \frac{\partial w}{\partial \theta} \sin \theta \right)$$

$$= \sin \theta \cos \theta \frac{\partial^2 w}{\partial r^2} + \frac{1}{r^2} \sin^2 \theta \frac{\partial w}{\partial \theta} + \frac{\cos^2 \theta}{r} \frac{\partial^2 w}{\partial r \partial \theta} - \frac{\sin \theta \cos \theta}{r} \frac{\partial w}{\partial r} - \frac{\sin \theta \cos \theta}{r^2} \frac{\partial^2 w}{\partial \theta^2} - \frac{\cos^2 \theta}{r^2} \frac{\partial w}{\partial \theta}$$

$$= \sin \theta \cos \theta \frac{\partial^2 w}{\partial r^2} - \frac{\cos 2\theta}{r^2} \frac{\partial w}{\partial \theta} + \frac{\cos 2\theta}{r} \frac{\partial^2 w}{\partial r \partial \theta} - \frac{\sin \theta \cos \theta}{r} \frac{\partial w}{\partial r}$$

Thus

$$- \frac{\sin \theta \cos \theta}{r^2} \frac{\partial^2 w}{\partial \theta^2}$$

$$\begin{aligned}
\left(\frac{\partial^2 \omega}{\partial x \partial y}\right)^2 &= \sin^2 \theta \cos^2 \theta \left(\frac{\partial^2 \varphi}{\partial n^2}\right)^2 + \frac{\cos^2 \theta}{n^2} \left(\frac{\partial^2 \varphi}{\partial t}\right)^2 + \frac{\cos^2 \theta}{n^2} \left(\frac{\partial^2 \varphi}{\partial t \partial \theta}\right)^2 \\
&+ \frac{\sin^2 \theta \cos^2 \theta}{n^2} \left(\frac{\partial^2 \varphi}{\partial z}\right)^2 + \frac{\sin^2 \theta \cos^2 \theta}{n^4} \left(\frac{\partial^2 \varphi}{\partial t^2}\right)^2 - \frac{\sin \theta \cos \theta}{n^2} \frac{\partial \varphi}{\partial \theta} \frac{\partial^2 \varphi}{\partial n^2} \\
&+ \frac{\sin \theta \cos \theta}{n} \frac{\partial \varphi}{\partial n^2} \frac{\partial^2 \varphi}{\partial z \partial \theta} - \frac{1}{2} \frac{\sin^2 \theta}{n} \frac{\partial \varphi}{\partial n} \frac{\partial^2 \varphi}{\partial n^2} - \frac{1}{2} \frac{\sin^2 \theta}{n^2} \frac{\partial^2 \varphi}{\partial n} \frac{\partial^2 \varphi}{\partial \theta^2} \\
&- 2 \frac{\cos^2 \theta}{n^3} \frac{\partial \varphi}{\partial \theta} \frac{\partial^2 \varphi}{\partial z \partial \theta} + \frac{\sin \theta \cos \theta}{n^3} \frac{\partial \varphi}{\partial n} \frac{\partial^2 \varphi}{\partial \theta} + \frac{\sin \theta \cos \theta}{n^4} \frac{\partial \varphi}{\partial \theta} \frac{\partial^2 \varphi}{\partial \theta^2} \\
&- \frac{\sin \theta \cos \theta}{n^2} \frac{\partial^2 \varphi}{\partial n} \frac{\partial^2 \varphi}{\partial z \partial \theta} - \frac{\sin \theta \cos \theta}{n^3} \frac{\partial^2 \varphi}{\partial n \partial \theta} \frac{\partial^2 \varphi}{\partial \theta^2} + \frac{1}{2} \frac{\cos^2 \theta}{n^3} \frac{\partial \varphi}{\partial n} \frac{\partial^2 \varphi}{\partial \theta^2}
\end{aligned}$$

$$\begin{aligned}
\left(\frac{\partial^2 \omega}{\partial x \partial y}\right)^2 - \frac{\partial^2 \omega}{\partial x^2} \frac{\partial^2 \omega}{\partial y^2} &= \frac{1}{n^4} \left(\frac{\partial \omega}{\partial t}\right)^2 + \frac{1}{n^2} \left(\frac{\partial^2 \varphi}{\partial t \partial \theta}\right)^2 - \frac{1}{n} \frac{\partial \omega}{\partial n} \frac{\partial^2 \omega}{\partial n^2} - \frac{1}{n^2} \frac{\partial^2 \omega}{\partial n} \frac{\partial^2 \omega}{\partial \theta^2} \\
&- 2 \frac{1}{n^3} \frac{\partial \omega}{\partial \theta} \frac{\partial^2 \omega}{\partial z \partial \theta}
\end{aligned}$$

We have $\left(\frac{1}{R}\right)_\theta = \frac{\left(\frac{a}{R}\right)^2 - \left(\frac{a}{R}\right)^2 \sin^2 \theta}{2}$

$$\left(\frac{1}{R}\right) = \frac{\left(\frac{a}{R}\right)^2 - \left(\frac{a}{R}\right)^2 \sin^2 \theta}{2} - f \left\{ \frac{\left(\frac{a}{R}\right)^2 - \left(\frac{a}{R}\right)^2}{2} \right\}$$

Thus

$$\begin{cases} \frac{1}{R} \frac{\partial u}{\partial \theta} = -\left(\frac{a}{R}\right)^2 \sin \theta \cos \theta = -\frac{1}{2} \left(\frac{a}{R}\right)^2 \sin 2\theta \\ \frac{1}{R} \frac{\partial u}{\partial \phi} = -\left(\frac{a}{R}\right)^2 \sin \theta \cos \theta = -\frac{1}{2} \left(\frac{a}{R}\right)^2 \sin 2\theta \end{cases}$$

$$\begin{cases} \frac{1}{R} \frac{\partial^2 u}{\partial \theta^2} = -\frac{1}{2} \left(\frac{a}{R}\right)^2 \sin 2\theta \\ \frac{1}{R} \frac{\partial^2 u}{\partial \theta \partial \phi} = -\frac{1}{2} \left(\frac{a}{R}\right)^2 \sin 2\theta \end{cases}$$

$$\begin{cases} \frac{1}{R} \frac{\partial^2 u}{\partial \theta^2} = -\left(\frac{a}{R}\right)^2 \cos 2\theta \\ \frac{1}{R} \frac{\partial^2 u}{\partial \phi^2} = -\left(\frac{a}{R}\right)^2 \cos 2\theta \end{cases}$$

$$\begin{cases} \frac{1}{R} \frac{\partial u}{\partial r} = -\frac{1}{2} \left(\frac{a}{R}\right)^2 \sin^2 \theta + f \frac{1}{R} \left(\frac{a}{R}\right)^2 \\ \frac{1}{R} \frac{\partial u}{\partial \phi} = -\frac{1}{2} \left(\frac{a}{R}\right)^2 \sin^2 \theta \end{cases}$$

$$\begin{cases} \frac{1}{R} \frac{\partial^2 u}{\partial r^2} = -\frac{1}{R^2} \sin^2 \theta + \frac{f}{R^2} \\ \frac{1}{R} \frac{\partial^2 u}{\partial r \partial \phi} = -\frac{1}{R^2} \sin^2 \theta \end{cases}$$

$$\frac{1}{r^2} \left(\frac{\partial u}{\partial \theta} \right)^2 - \frac{1}{r^2} \left(\frac{\partial u}{\partial \phi} \right)^2 = 0$$

$$\frac{1}{r^2} \left(\frac{\partial^2 u}{\partial r^2} \right)^2 - \frac{1}{r^2} \left(\frac{\partial^2 u}{\partial r \partial \theta} \right)^2 = 0$$

$$- \left\{ \frac{1}{r} \frac{\partial u}{\partial r} \frac{\partial^2 u}{\partial r^2} - \frac{1}{r} \frac{\partial u}{\partial \theta} \frac{\partial^2 u}{\partial r \partial \theta} \right\} = \left(\frac{1}{R^2} \sin^2 \theta \right)^2 \left[\frac{1}{R^2} \sin^2 \theta - \frac{f}{R^2} \right]^2$$

$$= \frac{1}{R^4} [2 \sin^2 \theta - f] f \cdot R^2$$

$$- \left\{ \frac{1}{r^2} \frac{\partial^2 u}{\partial r^2} \frac{\partial^2 u}{\partial \theta^2} - \frac{1}{r^2} \frac{\partial^2 u}{\partial r \partial \theta} \frac{\partial^2 u}{\partial r \partial \theta} \right\} = \frac{1}{R^2} \cos 2\theta \left\{ \frac{f}{R^2} \right\} = \frac{1}{R^4} + \cos 2\theta \cdot R^2$$

$$- 2 \left\{ \frac{1}{r^3} \frac{\partial u}{\partial \theta} \frac{\partial^2 u}{\partial r \partial \theta} \right\} = \frac{1}{R^2} \sin 2\theta \left\{ - \right\} = 0$$

The equation for the stream function ψ is simply

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \left(\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} \right) = \frac{Ef}{R^2} \left\{ \cos 2\theta + 2 \sin^2 \theta - f \right\}$$

$$= \frac{Ef(1-f)}{R^2} = K$$

First we have to find the particular integral of the equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \left(\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} \right) = K$$

Assuming ψ independent of θ ,

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) \left(\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} \right) = K$$

Let $\phi = c r^p$

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$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} = [p(p-1) + p] r^{p-2} = c p^2 r^{p-2}$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) \left(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} \right) = c p^2 (p-2)^2 r^{p-4} = K$$

$$\therefore \underline{p=4}$$

Also $\phi = c r^4$

$$c \cdot p^2 (p-2)^2 = K$$

$$c = \frac{K}{16 \cdot 4} = \underline{\underline{\frac{K}{64}}}$$

Hence the particular integral is

$$\boxed{\phi = \frac{K}{64} r^4}$$

Due to the symmetry of the problem, the solution of the homogeneous equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \left(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} \right) = 0$$

can be written as

$$\phi = \sum_{n=1}^{\infty} r^{2n} (A_n \cos 2(n-1)\theta + B_n \cos 2n\theta) \quad \dots (2)$$

Hence the complete solution is

$$\phi = \frac{K}{64} r^4 + \sum_{n=1}^{\infty} r^{2n} (A_n \cos 2(n-1)\theta + B_n \cos 2n\theta) \dots (3)$$

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$$\begin{aligned}\sigma_r &= \frac{K}{16} r^2 - \sum_1 r^{2(n-1)} \left[A_n \left\{ 2n - \frac{1}{4}(n-1)^2 \right\} \cos \frac{1}{2}(n-1)\theta + b_n \left\{ 2n - 4n^2 \right\} \cos 2n\theta \right] \\ \sigma_\theta &= \frac{3K}{16} r^2 + \sum_1 2n(2n-1) r^{2(n-1)} \left[A_n \cos \frac{1}{2}(n-1)\theta + b_n \cos 2n\theta \right] \\ \tau_{r\theta} &= \sum_1 (2n-1) r^{2(n-1)} \left[\frac{1}{2}(2n-1) A_n \sin \frac{1}{2}(n-1)\theta + 2n b_n \sin 2n\theta \right]\end{aligned}$$

The boundary conditions are at $r=a$, $\sigma_r = \tau_{r\theta} = 0$, $\sigma_\theta = \sigma(1-2\cos 2\theta)$
Thus

$$0 = \frac{K}{16} a^2 + \sum_1 \left[\left\{ 2n - \frac{1}{4}(n-1)^2 \right\} A_n \cos \frac{1}{2}(n-1)\theta + \left\{ 2n - 4n^2 \right\} b_n \cos 2n\theta \right]$$

$$\sigma(1-2\cos 2\theta) = \frac{3K}{16} r^2 + \sum_1 2n(2n-1) \left[A_n \cos \frac{1}{2}(n-1)\theta + b_n \cos 2n\theta \right]$$

$$0 = \sum_1 (2n-1) \left[\frac{1}{2}(2n-1) A_n \sin \frac{1}{2}(n-1)\theta + 2n b_n \sin 2n\theta \right]$$

where $\underline{a_n = r^{(n-1)} A_n}$

$$\therefore \frac{E\mu(1-f)}{16} \left(\frac{a}{r_0}\right)^2 = -2a_1, \quad 0 = \left\{ \frac{1}{2}(2n+1) - 4n^2 \right\} a_{n+1} + \left\{ 2n - 4n^2 \right\} b_n$$

$$\sigma = \frac{3E\mu(1-f)}{16} \left(\frac{a}{r}\right)^2 = 2a_1, \quad -2\sigma = 2b_1 + 12a_2,$$

$$2(n+1) \left[\frac{1}{2}(2n+1) \right] a_{n+1} + 2n(2n-1) b_n = 0$$

$$(2n+1)2n a_{n+1} + 2n(2n-1)b_n = 0$$

It is then necessary to satisfy the stress free boundary condition

$$\begin{aligned} \text{But } \sigma_r &= E \frac{\partial u}{\partial r} + \frac{E}{2} \left\{ \frac{\partial^2 v}{\partial \theta^2} - \frac{\partial^2 u}{\partial \theta^2} \right\} \\ &= E \frac{\partial^2 u}{\partial r^2} + \frac{E}{2} f\left(\frac{a}{R}\right) \left[f\left(\frac{a}{R}\right) - 2f\left(\frac{a}{R}\right) \sin^2 \theta \right] \\ &= E \frac{\partial^2 u}{\partial r^2} + \frac{E}{2} \frac{2(1-f)(\frac{a}{R})^2}{2} + \frac{E}{2} f\left(\frac{a}{R}\right)^2 \cos 2\theta \\ \sigma_\theta &= E \left\{ \frac{u}{r} + \frac{\partial^2 v}{\partial \theta^2} \right\} \end{aligned}$$

$$\tau_{r\theta} = \frac{E}{2} \left\{ \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial^2 v}{\partial r \partial \theta} - \frac{1}{r} \right\} - \frac{E}{2} \frac{1}{r} \left(\frac{a}{R} \right) \sin 2\theta f\left(\frac{a}{R}\right)$$

Hence

$$\begin{aligned} E \frac{\partial^2 u}{\partial r^2} &= \frac{9}{16} E f(1-f) \left(\frac{a}{R}\right)^2 - \frac{1}{2} E f \left(\frac{a}{R}\right)^2 \cos 2\theta \\ &+ \sum \left(\frac{a}{R}\right)^{2(n-1)} \left[A_n \{2n-4n^2\} \cos 2(n-1)\theta \right. \\ &\quad \left. + B_n \{2n-4n^2\} \cos 2n\theta \right] \end{aligned}$$

$$E \left[\frac{u}{r} + \frac{\partial^2 v}{\partial \theta^2} \right] = \frac{3E f(1-f)}{16} \left(\frac{a}{R}\right)^2 + \sum 2n(2n-1) \left(\frac{a}{R}\right)^{2n-1} \left[A_n \cos 2(n-1)\theta + B_n \cos 2n\theta \right]$$

$$E \left[\frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial^2 v}{\partial r \partial \theta} - \frac{u}{r} \right] = \frac{1}{2} f \left(\frac{a}{R}\right)^2 \sin 2\theta$$

$$+ \sum 2(2n-1) \left(\frac{a}{R}\right)^{2n-1} \left[2(n-1) A_n \sin 2(n-1)\theta + 2n B_n \sin 2n\theta \right]$$

$$E \frac{u}{R} = \frac{3}{16} E f(1-f) \left(\frac{a}{R}\right)^3 - \frac{1}{6} E f \left(\frac{a}{R}\right)^3 \cos 2\theta$$

$$+ \sum \frac{1}{2n-1} \left(\frac{a}{R}\right)^{2n-1} \left[A_n \{2n-4(n-1)\} \cos 2(n-1)\theta + B_n \{2n-4n\} \cos 2n\theta \right]$$

$$E \frac{u}{R} = \frac{3}{16} E f(1-f) \left(\frac{a}{R}\right)^3 - \frac{1}{6} E f \left(\frac{a}{R}\right)^3 \cos 2\theta$$

$$+ \sum \left(\frac{a}{R}\right)^{2n-1} \left[A_n (4-2n) \cos 2(n-1)\theta - B_n 2n \cos 2n\theta \right]$$

$$E \frac{u}{R} = \frac{3}{16} E f(1-f) \left(\frac{a}{R}\right)^2 - \frac{1}{6} E f \left(\frac{a}{R}\right)^2 \cos 2\theta$$

$$+ \sum \left(\frac{a}{R}\right)^{2(n-1)} \left[(4-2n) A_n \cos 2(n-1)\theta - 2n B_n \cos 2n\theta \right]$$

$$E \frac{1}{11} \frac{u}{R} = \frac{1}{6} E f \left(\frac{a}{R}\right)^3 \cos 2\theta + \sum \left(\frac{a}{R}\right)^{2(n-1)} \left[4(n^2-1) A_n \cos 2(n-1)\theta + 4n^2 B_n \cos 2n\theta \right]$$

$$E \frac{v}{R} = \frac{1}{12} E f \left(\frac{a}{R}\right)^3 \sin 2\theta + \sum \left(\frac{a}{R}\right)^{2n-1} \left[-2(n+1) A_n \sin 2(n+1)\theta + 2n B_n \sin 2n\theta \right]$$

But from page 161

$$E \frac{u}{R} = \sigma \left(\frac{a}{R}\right) [1 + 2 \cos 2\theta]$$

$$E \frac{v}{R} = -\sigma \left(\frac{a}{R}\right) \sin 2\theta$$

$$\sigma \frac{a}{R} = \frac{3}{16} E f (1-f) \left(\frac{a}{R}\right)^3 + \left(\frac{a}{R}\right) \cdot 2A_1$$

$$2\sigma \left(\frac{a}{R}\right) = -\frac{1}{6} E f \left(\frac{a}{R}\right)^3 - 2\left(\frac{a}{R}\right) b_1$$

$$-\sigma \left(\frac{a}{R}\right) = \frac{1}{12} E f \left(\frac{a}{R}\right)^3 + 2\left(\frac{a}{R}\right) b_1$$

Three equations for three unknowns f, A_1, b_1

$$\frac{\sigma}{E} = -\frac{1}{12} f \left(\frac{a}{R}\right)^3$$

 \therefore

$$f = \frac{-12\sigma}{E \left(\frac{a}{R}\right)^3}$$

$$b_1 = -\frac{1}{6} E f \left(\frac{a}{R}\right)^3 - \sigma$$

$$= 2\sigma - \sigma = \sigma$$

$$b_1 = \sigma$$

$$\therefore \sigma = \frac{3}{16} E f (1-f) \left(\frac{a}{R}\right)^3 + 2A_1$$

$$A_1 = \frac{\sigma}{2} - \frac{3}{32} (-12\sigma) \left[1 + \frac{12\sigma}{E \left(\frac{a}{R}\right)^3} \right]$$

$$A_1 = \frac{\sigma}{2} + \frac{9}{8} \sigma \left[1 + \frac{12\sigma}{E \left(\frac{a}{R}\right)^3} \right]$$

$$\left(\frac{\sigma}{R^2}\right) = \underbrace{\frac{E t (1-\nu)}{6t} \left(\frac{A}{R}\right)^4}_{C_1} + \underbrace{\left(\frac{A}{R}\right)^2 \left[\frac{\sigma}{2} - \frac{3}{32} E t (1-\nu) \left(\frac{A}{R}\right)^2 \right]}_{C_2} + \underbrace{\left(\frac{A}{R}\right)^2 \sigma \cos 2\theta}_{C_3}$$

$$\text{or } \frac{\sigma}{R^2} = C_1 \left(\frac{A}{R}\right)^4 + C_2 \left(\frac{A}{R}\right)^2 + C_3 \left(\frac{A}{R}\right)^2 \cos 2\theta$$

$$\sigma_r = 4C_1 \left(\frac{A}{R}\right)^2 + 2C_2 + 2C_3 \cos 2\theta - 4C_3 \sin 2\theta$$

$$\sigma_\theta = 12C_1 \left(\frac{A}{R}\right)^2 + 2C_2 + 2C_3 \cos 2\theta$$

$$r_{\theta\theta} = + 2C_3 \sin 2\theta$$

The extensional energy

$$\begin{aligned} & 4 \frac{E t}{2} \frac{1}{E^2} R^2 \int_0^{a/R} \int_0^{2\pi} \left\{ \left[2C_1 \left(\frac{A}{R}\right)^2 + C_2 - C_3 \cos 2\theta \right]^2 \right. \\ & \quad \left. + \left[6C_1 \left(\frac{A}{R}\right)^2 + C_2 + C_3 \cos 2\theta \right]^2 + 2C_3^2 \sin^2 2\theta \right\} \left(\frac{A}{R}\right) d\left(\frac{A}{R}\right) d\theta \\ &= \frac{2t R^2 \pi}{E} \int_0^{a/R} \left[8C_1^2 \left(\frac{A}{R}\right)^4 + 2C_2^2 + C_3^2 + 72C_1^2 \left(\frac{A}{R}\right)^4 + 2C_2^2 + C_3^2 + 2C_3^2 \left(\frac{A}{R}\right)^4 \right] d\left(\frac{A}{R}\right) \\ &= \frac{2t R^2 \pi}{E} \int_0^{a/R} \left[80C_1^2 \left(\frac{A}{R}\right)^5 + 4(C_2^2 + C_3^2) \frac{A^2}{R} \right] d\left(\frac{A}{R}\right) \end{aligned}$$

$$= \frac{8tR^2\pi}{E} \left\{ \frac{10}{3} C_1^2 \left(\frac{a}{R}\right)^6 + \frac{1}{2} (C_2^2 + C_3^2) \left(\frac{a}{R}\right)^2 \right\}$$

The bending energy

$$\frac{1}{R} \left[\frac{\partial^2 w}{\partial \theta^2} + \frac{1}{2} \frac{\partial w}{\partial \theta} + \frac{1}{2} \frac{\partial^2 w}{\partial \theta^2} \right] - \frac{1}{R} \left[\frac{\partial^2 w}{\partial \theta^2} + \frac{1}{2} \frac{\partial w}{\partial \theta} + \frac{1}{2} \frac{\partial^2 w}{\partial \theta^2} \right]$$

$$= \frac{1}{R^2} \left[\right]$$

In case of pure rotation $\omega = 0$

$$k_1 = \frac{\partial^2 w}{\partial \theta^2} - \frac{\partial^2 w}{\partial \theta^2}$$

$$k_2 = \frac{1}{2} \frac{\partial^2 w}{\partial \theta^2} + \frac{1}{2} \frac{\partial w}{\partial \theta} - \frac{1}{2} \frac{\partial^2 w}{\partial \theta^2} - \frac{1}{2} \frac{\partial w}{\partial \theta}$$

$$\tau = \frac{\partial}{\partial \theta} \left(\frac{1}{2} \frac{\partial w}{\partial \theta} \right) - \left(\frac{\partial}{\partial \theta} \cdot \frac{1}{2} \frac{\partial w}{\partial \theta} \right)$$

$$\frac{1}{R} k_1 = \frac{f}{R^2}$$

$$\tau = 0$$

$$\frac{1}{R} k_2 = \frac{f}{R^2}$$

$$\text{Bending energy} = \frac{Et}{2} \frac{t^3}{12} 2 \left(\frac{f}{R}\right)^2 \pi a^2 = \frac{E}{12} \left(\frac{t}{R}\right)^3 \pi \left(\frac{a}{R}\right)^2 f^2 R^3$$

Total energy

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$$\frac{W}{R^3} = \frac{f(\frac{t}{R})}{E} \pi (\frac{a}{R})^2 \left\{ \frac{10}{3} C_1^2 (\frac{a}{R})^4 + \frac{1}{2} (C_2^2 + C_3^2) \right\} + \frac{E}{12} (\frac{t}{R})^3 f^2 \pi (\frac{a}{R})^2$$

$$= (\frac{t}{R}) \pi (\frac{a}{R})^2 \left[\frac{8}{E} \left\{ \frac{10}{3} C_1^2 (\frac{a}{R})^4 + \frac{1}{2} (C_2^2 + C_3^2) \right\} + \frac{E}{12} (\frac{t}{R})^2 f^2 \right]$$

Decrease in energy

$$\frac{\Delta W}{R^3} = (\frac{t}{R}) \pi (\frac{a}{R})^2 \left[\frac{c^2}{2E} - \frac{8}{E} \left\{ \frac{10}{3} C_1^2 (\frac{a}{R})^4 + \frac{1}{2} (C_2^2 + C_3^2) \right\} - \frac{E}{12} (\frac{t}{R})^2 f^2 \right]$$

Total decrease in energy

$$\frac{\Delta W}{R^3} = \frac{t}{R} \pi \frac{a^2}{R} \left[\frac{9}{32} \frac{c^2}{E} - \frac{8}{E} \left\{ \frac{10}{3} C_1^2 (\frac{a}{R})^4 + \frac{1}{2} (C_2^2 + C_3^2) \right\} - \frac{E}{12} (\frac{t}{R})^2 f^2 \right]$$

On substituting it explicitly, we have

$$\frac{\Delta W}{R^3} = (\frac{t}{R}) \pi (\frac{a}{R})^2 \left[\frac{9}{32} \frac{c^2}{E} - \frac{8}{E} \left\{ \frac{15}{128} \sigma^2 \left[1 + \frac{12\sigma}{E(\frac{a}{R})^2} \right]^2 + \frac{1}{2} \left[\sigma^2 + \left\{ \frac{5}{2} + \frac{9}{8} \sigma \left(1 + \frac{12\sigma}{E(\frac{a}{R})^2} \right) \right\}^2 \right] \right\} - \frac{E}{12} (\frac{t}{R})^2 \frac{144\sigma^2}{E^2 (\frac{a}{R})^4} \right]$$

$$= (\frac{t}{R}) \pi (\frac{a}{R})^2 \frac{\pi^2}{E} \left[\frac{9}{32} - \frac{15}{16} \left\{ 1 + \frac{12\sigma}{E(\frac{a}{R})^2} \right\}^2 - 5 - \frac{41}{16} \left\{ 1 + \frac{12\sigma}{E(\frac{a}{R})^2} \right\}^2 - \frac{9}{2} \left\{ 1 + \frac{12\sigma}{E(\frac{a}{R})^2} \right\} - 12 (\frac{t}{R})^2 \frac{1}{(\frac{a}{R})^4} \right]$$

$$\frac{\Delta W}{R^3} = \left(\frac{t}{R}\right) \pi \left(\frac{a}{R}\right)^2 \frac{\sigma^2}{E} \left[\frac{9}{32} - 5 - 6 \left\{ 1 + \frac{12\sigma}{E \left(\frac{a}{R}\right)^2} \right\}^2 - \frac{9}{2} \left\{ 1 + \frac{12\sigma}{E \left(\frac{a}{R}\right)^2} \right\} - 12 \left(\frac{t}{R}\right)^2 \frac{1}{\left(\frac{a}{R}\right)^4} \right]$$

$$= \left(\frac{t}{R}\right) \pi \left(\frac{a}{R}\right)^2 \frac{\sigma^2}{E} \left[-\frac{417}{32} - 198 \frac{\sigma}{E \left(\frac{a}{R}\right)^2} - 864 \left(\frac{\sigma}{E}\right)^2 \frac{1}{\left(\frac{a}{R}\right)^4} - 12 \left(\frac{t}{R}\right)^2 \frac{1}{\left(\frac{a}{R}\right)^4} \right]$$

Pulling $\sigma = -\sigma$

$$\frac{\Delta W}{R^3} = \left(\frac{t}{R}\right) \pi \frac{\sigma^2}{E} \left[198 \left(\frac{\sigma}{E}\right) - \frac{417}{32} \left(\frac{a}{R}\right)^2 - 864 \left(\frac{\sigma}{E}\right)^2 \frac{1}{\left(\frac{a}{R}\right)^2} - 12 \left(\frac{t}{E}\right)^2 \left(\frac{E \left(\frac{t}{R}\right)}{\sigma}\right)^2 \frac{1}{\left(\frac{a}{R}\right)^4} \right]$$

$$\frac{417}{32} \left(\frac{a}{R}\right)^2 = 198 \left(\frac{\sigma}{E}\right)^2 \left[72 + \left(\frac{E \left(\frac{t}{R}\right)}{\sigma}\right)^2 \right]$$

$$\therefore \left(\frac{a}{R}\right)^2 = \sqrt{\frac{324}{417}} \frac{\sigma}{E} \left[72 + \left(\frac{E \left(\frac{t}{R}\right)}{\sigma}\right)^2 \right]^{\frac{1}{2}}$$

$$\frac{\Delta W}{R^3} = \left(\frac{t}{R}\right) \pi \frac{\sigma^2}{E} \left[198 \left(\frac{\sigma}{E}\right) - 2 \sqrt{\frac{417 \times 12}{32}} \left(\frac{\sigma}{E}\right) \left\{ 72 + \left(\frac{E \left(\frac{t}{R}\right)}{\sigma}\right)^2 \right\}^{\frac{1}{2}} \right]$$

$$= \left(\frac{t}{R}\right) \pi \frac{\sigma^2}{E} \left(\frac{\sigma}{E}\right) \left[198 - 2 \sqrt{\frac{1461}{8}} \left\{ 72 + \left(\frac{E \left(\frac{t}{R}\right)}{\sigma}\right)^2 \right\} \right]$$

Thus at σ_c , $\Delta W = 0$

$$198^2 = \frac{1461}{8} \left\{ 72 + \left(\frac{E \left(\frac{t}{R}\right)}{\sigma}\right)^2 \right\}$$

$$\sqrt{\frac{198^2}{1461} - 66} = \left(\frac{E \left(\frac{t}{R} \right)}{\sigma} \right) = 12.19$$

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100
300000

$$\frac{214.66}{66} = 148.66$$

$$\sigma_{ci} = 0.0121 E \left(\frac{t}{R} \right)$$

at σ_{ci}

$$198 \sqrt{\frac{32}{487 \times 12}} = \sqrt{66 + \left(\frac{E \left(\frac{t}{R} \right)}{\sigma} \right)^2}$$

$$\left(\frac{a}{R} \right)^2 = \frac{32}{487} \frac{\sigma}{E} = \frac{32}{487} \times 0.0121 \left(\frac{t}{R} \right)$$

$$\left(\frac{a}{R} \right) = 0.0735 \sqrt{\left(\frac{t}{R} \right)}$$

The residual stress at the edges are

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$$\sigma_r = 4C_1 \left(\frac{a}{R}\right)^2 + 2C_2 - 2C_3 \cos 2\theta$$

$$\sigma_r = \frac{E\ell(1-\ell)}{16} \left(\frac{a}{R}\right)^2 + \left\{ \sigma - \frac{3}{16} E\ell(1-\ell) \left(\frac{a}{R}\right)^2 \right\} - 2\sigma \cos 2\theta$$

$$\sigma_\theta = \frac{3E\ell(1-\ell)}{16} \left(\frac{a}{R}\right)^2 + \left\{ \sigma - \frac{3}{16} E\ell(1-\ell) \left(\frac{a}{R}\right)^2 \right\} + 2\sigma \cos 2\theta$$

$$\tau_{r\theta} = 2\sigma \sin 2\theta$$

The additional stress system is

$$\begin{cases} \sigma_r = -\frac{E\ell(1-\ell)}{8} \left(\frac{a}{R}\right)^2 + \sigma(1-2\cos 2\theta) \\ \sigma_\theta = 4\sigma \cos 2\theta \\ \tau_{r\theta} = 2\sigma \sin 2\theta \end{cases}$$

[See p. 315, Sn. 11.10 "elasticity", Eq. 11.11 & 11.12]

$$P_0 = \sigma - \frac{E\ell(1-\ell)}{8} \left(\frac{a}{R}\right)^2$$

$$\begin{aligned} 2\sigma &= \sigma_1 + \sigma_2 \\ -2\sigma &= \sigma_1 - \sigma_2 \end{aligned} \quad \left. \begin{aligned} 4\sigma &= 2\sigma_2 \\ \sigma_2 &= 2\sigma \\ \sigma_1 &= -\sigma \end{aligned} \right\}$$

Notice above that only the σ_r & $\tau_{r\theta}$ conditions are satisfied ²⁴

$$\sigma_r = \left\{ \sigma - \frac{E t (1-\nu)}{8} \left(\frac{\sigma}{R} \right)^2 \right\} \left(\frac{a}{r} \right)^2 + \left\{ 6\sigma \left(\frac{a}{r} \right)^4 - 8\sigma \left(\frac{a}{r} \right)^2 \right\} \cos 2\theta$$

$$\tau_{r\theta} = - \left\{ \sigma - \frac{E t (1-\nu)}{8} \left(\frac{\sigma}{R} \right)^2 \right\} \left(\frac{a}{r} \right)^2 - 6\sigma \left(\frac{a}{r} \right)^4 \cos 2\theta$$

$$\tau_{\theta\theta} = \left\{ 6\sigma \left(\frac{a}{r} \right)^4 - 4\sigma \left(\frac{a}{r} \right)^2 \right\} \sin 2\theta$$

$$\pi a^2 \frac{t}{2E} \int_0^\pi \left[2 \left\{ \sigma - \frac{E t (1-\nu)}{8} \left(\frac{\sigma}{R} \right)^2 \right\}^2 \frac{1}{\left(\frac{a}{r} \right)^3} + \left\{ 6\sigma \frac{1}{\left(\frac{a}{r} \right)^4} - 8\sigma \frac{1}{\left(\frac{a}{r} \right)^2} \right\} \left(\frac{a}{r} \right)^2 \right. \\$$

$$\left. + 2 \left\{ \sigma - \frac{E t (1-\nu)}{8} \left(\frac{\sigma}{R} \right)^2 \right\}^2 \frac{1}{\left(\frac{a}{r} \right)^3} + 36\sigma^2 \frac{1}{\left(\frac{a}{r} \right)^7} \right. \\$$

$$\left. + 2 \left\{ 6\sigma \frac{1}{\left(\frac{a}{r} \right)^4} - 4\sigma \frac{1}{\left(\frac{a}{r} \right)^2} \right\}^2 \left(\frac{a}{r} \right)^2 \right] \frac{1}{r} \left(\frac{a}{r} \right) dr$$

$$= \frac{\pi a^2 t \sigma^2}{2E} \left[2 \left\{ 1 - \frac{E t (1-\nu)}{8\sigma} \left(\frac{a}{R} \right)^2 \right\}^2 + (6 - 24 + 32 + 6 + 12 - 24 + 16) \right]$$

$$= \frac{\pi a^2 t \sigma^2}{E} \left\{ \left[1 - \frac{E t (1-\nu)}{8\sigma} \left(\frac{a}{R} \right)^2 \right]^2 + 12 \right\}$$

$$\frac{\Delta W_1}{R^3} = \left(\frac{t}{R}\right) \pi \frac{\sigma^2}{E} \left\{ 13 \left(\frac{a}{R}\right)^2 - \frac{E f (1-f)}{4\sigma} \left(\frac{a}{R}\right)^4 + \frac{E^2 f^2 (1-f)^2}{64\sigma^2} \left(\frac{a}{R}\right)^6 \right\} \quad \underline{\underline{46}}$$

$$= \left(\frac{t}{R}\right) \pi \frac{\sigma^2}{E} \left\{ 13 \left(\frac{a}{R}\right)^2 + 3 \left(1 + \frac{12\sigma}{E \left(\frac{a}{R}\right)^2}\right) \left(\frac{a}{R}\right)^4 \right. \\ \left. + \frac{9}{4} \left(1 + \frac{12\sigma}{E \left(\frac{a}{R}\right)^2}\right)^2 \left(\frac{a}{R}\right)^6 \right\}$$

$$= \left(\frac{t}{R}\right) \pi \frac{\sigma^2}{E} \left\{ 13 \left(\frac{a}{R}\right)^2 + 3 \left(\frac{a}{R}\right)^4 + \frac{9}{4} \left(\frac{a}{R}\right)^6 + \frac{36\sigma}{E} \left(\frac{a}{R}\right)^2 \right. \\ \left. + \frac{54\sigma}{E} \left(\frac{a}{R}\right)^4 + 9 \times 36 \left(\frac{\sigma}{E}\right)^2 \left(\frac{a}{R}\right)^2 \right\}$$

$$\frac{\Delta W_1}{R^3} = \left(\frac{t}{R}\right) \pi \frac{\sigma^2}{E} \left\{ \left[13 - 36 \left(\frac{\sigma}{E}\right) + 324 \left(\frac{\sigma}{E}\right)^2\right] \left(\frac{a}{R}\right)^2 + \left[3 - 54 \left(\frac{\sigma}{E}\right)\right] \left(\frac{a}{R}\right)^4 \right. \\ \left. + \frac{9}{4} \left(\frac{a}{R}\right)^6 \right\}$$

$$\frac{\Delta W}{R^3} = \left(\frac{t}{R}\right) \pi \frac{\sigma^2}{E} \left\{ 198 \left(\frac{\sigma}{E}\right) - \left[\frac{903}{32} - 36 \left(\frac{\sigma}{E}\right) + 324 \left(\frac{\sigma}{E}\right)^2\right] \left(\frac{a}{R}\right)^2 - \left[3 - 54 \left(\frac{\sigma}{E}\right)\right] \left(\frac{a}{R}\right)^4 \right. \\ \left. - \frac{9}{4} \left(\frac{a}{R}\right)^6 - 12 \left(\frac{\sigma}{E}\right)^2 \left[42 + \left(\frac{E t}{\sigma R}\right)^2\right] \frac{1}{\left(\frac{a}{R}\right)^2} \right\}$$

Recalculation of Strain energy increase due to the presence of a hole:

$$\hat{n}_r = \frac{1}{2}\sigma \left(1 - \frac{a^2}{r^2}\right) \left[1 + \left(1 - 3\frac{a^2}{r^2}\right) \cos 2\theta\right]$$

$$\hat{\theta}_r = \frac{1}{2}\sigma \left[\left(1 + \frac{a^2}{r^2}\right) - \left(1 + \frac{3a^4}{r^4}\right) \cos 2\theta \right]$$

$$\hat{n}_\theta = -\frac{1}{2}\sigma \left(1 - \frac{a^2}{r^2}\right) \left(1 + 3\frac{a^2}{r^2}\right) \sin 2\theta$$

$$\hat{n}_0 = \frac{1}{2}\sigma (1 + \cos 2\theta)$$

$$\hat{\theta}_0 = \frac{1}{2}\sigma (1 - \cos 2\theta)$$

$$\hat{n}_\theta = -\frac{1}{2}\sigma \sin 2\theta$$

Strain energy increase due to the presence of a hole

$$= \frac{\pi a^2 t \sigma^2}{8E} \int_1^\infty \left(\frac{1}{a}\right) d\left(\frac{1}{a}\right) \left[\left(1 - \frac{a^2}{r^2}\right)^2 \left\{ 2 + \left(1 - 3\frac{a^2}{r^2}\right)^2 \right\} + 2\left(1 + \frac{a^2}{r^2}\right)^2 + \left(1 + \frac{3a^4}{r^4}\right)^2 + 2\left(1 - \frac{a^2}{r^2}\right)^2 \left(1 + 3\frac{a^2}{r^2}\right)^2 - 8 \right]$$

$$= \frac{\pi a^2 t \sigma^2}{8E} \int_1^\infty \left(\frac{1}{a}\right) d\left(\frac{1}{a}\right) \left[\left(1 - 2\frac{a^2}{r^2} + \frac{a^4}{r^4}\right) \left(3 - 6\frac{a^2}{r^2} + 9\frac{a^4}{r^4}\right) + 2\left(1 + 2\frac{a^2}{r^2} + \frac{a^4}{r^4}\right) + \left(1 + 6\frac{a^2}{r^2} + 9\frac{a^4}{r^4}\right) + 2\left(1 + 4\frac{a^2}{r^2} - 2\frac{a^4}{r^4} - 12\frac{a^6}{r^6} + 9\frac{a^8}{r^8}\right) - 8 \right]$$

$$= \frac{\pi a^2 t \sigma^2}{2E} \int_1^\infty \left(\frac{1}{a}\right) d\left(\frac{1}{a}\right) \left[7\left(\frac{a}{r}\right)^4 - 12\left(\frac{a}{r}\right)^6 + 9\left(\frac{a}{r}\right)^8 \right] = \frac{\pi a^2 t \sigma^2}{2E} \left[\frac{1}{2} + \frac{9}{6} \right]$$

$$\therefore \boxed{\frac{\Delta W_1}{R^3} = \pi \left(\frac{a}{R}\right)^2 \left(\frac{t}{R}\right) \frac{\sigma^2}{E} \left(\frac{t}{R}\right)}$$

Referring to page 24 Impossible!!!